ARTICLE IN PRESS YJABR:16744

Journal of Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Reflective Lorentzian lattices of signature $(5, 1)$

Ivica Turkalj

ARTICLE INFO

Article history:
Received 5 October 2017
Available online xxxx
Communicated by Gunter Malle

Keywords:
Integral quadratic forms
Integral lattices
Reflective lattices
Hyperbolic reflection groups
Vinberg algorithm

ABSTRACT

In this paper we give a complete classification of strongly square-free reflective $\mathbb{Z}$-lattices of signature $(5, 1)$. This is done by reducing the classification of Lorentzian lattices to those of positive-definite lattices. The classification of totally-reflective genera breaks up into two parts. The first part consists of classifying the square free, totally-reflective, primitive genera by calculating strong bounds on the prime factors of the determinant of positive-definite quadratic forms (lattices) with this property. We achieve these bounds by combining the Minkowski–Siegel mass formula with the combinatorial classification of reflective lattices accomplished by Scharlau & Blaschke. In a second part, we use a lattice transformation that goes back to Watson, to generate all totally-reflective, primitive genera when starting from the square free case.

1. Introduction

With this work, we wish to contribute to the problem of classifying arithmetic reflection groups on the hyperbolic space. More precisely, we analyze the connection between reflective Lorentzian lattices and maximal arithmetic reflection groups with non-compact fundamental domain. We refer to [3] for a detailed survey of known results and an overview of the historical development. Here, we only want to mention that contribu-

E-mail address: ivica.turkalj@math.tu-dortmund.de.

https://doi.org/10.1016/j.jalgebra.2018.06.013

Please cite this article in press as: I. Turkalj, Reflective Lorentzian lattices of signature (5, 1), J. Algebra (2018), https://doi.org/10.1016/j.jalgebra.2018.06.013

tions to this topic can be found in [31], [30], [34] [32], [17], [18], [33], [23], [27], [7], [10], [19], [20], [21], [1].

To sketch the notion of arithmeticity, consider a totally real number field F and its ring of integers $\mathfrak{o}_F$. Let E be a $\mathfrak{o}_F$-lattice of signature $(n, 1)$ such that for every non-identity embedding $\sigma : F \longrightarrow \mathbb{R}$ the lattice ${}^{\sigma}E$ is positive-definite. The isometry group $O^+(E)$ can be considered as a discrete subgroup of the full isometry group of the hyperbolic space of dimension n. Finite-index subgroups of groups obtained in this manner are called *arithmetic*. By definition, the group $O^+(E)$ is always arithmetic. If it is generated up to finite index by reflections, then the lattice E is called *reflective*.

Using arithmetic theory of quadratic forms and their connection to root systems, we reduced the classification of Lorentzian lattices to those of totally-reflective genera by proving that a strongly square-free reflective lattice E of signature $(5, 1)$ can always be written as $E = {}^{\alpha}\mathbb{H} \perp L$, with L totally-reflective and ${}^{\alpha}\mathbb{H} \cong \left(\begin{smallmatrix} 0 & \alpha \\ \alpha & 0 \end{smallmatrix}\right)$. Here, a positive-definite lattice is called totally-reflective if every lattice in its genus has a root system of full rank. We should mention that we are also interested in totally-reflective lattices as objects in their own right, thus the following result is more general than needed for the application to reflective lattices of signature $(5, 1)$. The strategy leading to the classification is as follows:

- Step 1: Let L be a strongly square-free totally-reflective lattice with $\dim L = 4$ (resp. $\dim = 3$). Hence $\det L$ is of the form $\det L = p_1^2 \cdots p_r^2 \cdot q_1 \cdots q_s$ (resp. $r = 0$). Using the mass formula and the combinatorial description of lattices with full-rank root system by Scharlau & Blaschke (cf. [25]), we prove that $r \leq 9$ and $s \leq 8 - r$ (resp. $s \leq 9$).
- Step 2: Then, we show that there are bounds c_i and d_j (one for every prime factor) depending only on the number of prime factors such that $p_i \leq c_i$ and $q_j \leq d_j$. Thus the number of local invariants that need to be taken into account is effectively bounded and the enumeration is computationally feasible.
- Step 3: After finishing the strongly square-free classification, we obtain all square-free, primitive totally-reflective genera by partial dualization.
- Step 4: The last step consists in dropping the assumption "square-free" by determining the pre-images of square-free genera under the Watson transformation.

It is worth mentioning that the primes which actually appear in the determinant are surprisingly low. The biggest prime is 23. After carrying out all the steps, we obtain the following result.

Theorem.

a) *In dimension* 3*, there are* 1234 *primitive totally-reflective genera of which* 289 *are square-free and* 52 *strongly square-free.*