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A unified closed-form expression of logit and weibit and its application to a transportation network equilibrium assignment

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Abstract

This study proposes a generalized multinomial logit model where heteroscedastic variance and flexible shape of utility function are allowed. The novel point of our approach is that, while the model is theoretically derived by applying a generalized extreme value distribution to the random component of utility, the model maintains its closed-form expression. Also, the weibit model, where the random utility is assumed to follow the Weibull distribution, is a special case of the proposed model. This is achieved by utilizing q-generalization method developed in Tsallis statistics. Then, the generalized logit model is incorporated into a transportation network equilibrium model. The network equilibrium model with the generalized logit route choice is formulated as an optimization problem under uncongested networks. The objective function includes Tsallis entropy, which is a type of generalized entropy. The generalization of the Gumbel and Weibull distributions, logit and weibit models, and network equilibrium model is made within a unified framework with q-analysis or Tsallis statistics.

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Keywords: closed-form expression; logit; weibit; Tsallis entropy; transportation network equiblium assignment

1. Introduction

The multinomial logit model plays an important role in transportation network analysis as well as travel behavior analysis. A stochastic user equilibrium model with a logit-based route choice (logit-based network equilibrium model) is one of the most widely used network equilibrium models. The multinomial logit model is closed-formed and is easily applicable. Such a simple closed formulation is desirable, especially considering the embedment of route choice model into network equilibrium analysis. The calculation of route choice probabilities is iterative, and

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requires significant computational costs in the network equilibrium models. The closed-form logit formulation is derived from the assumption of independent identical Gumbel distributions for error terms.

Castillo *et al.* (2008) proposed a closed-form discrete choice model with the Weibull-distributed utility. This is called weibit model. Fosgerau & Bierlaire (2009) considered a multiplicative error term, and derived a closed-form model similar to the weibit model of Castillo *et al.* (2008). Li (2011) extended the logit and weibit models for other distributions, and offered other alternative error distributions for discrete choice models. Kitthamkesorn & Chen (2013, 2014) proposed a stochastic user equilibrium model with the weibit route choice. The weibit model considers heterogeneous perceived variances with respect to different travel costs, while the (multinomial) logit model has the property of homogeneity in the variance of the error terms. Bhat (1995, 1997), DeShazo & Fermo (2002), Caussade et al. (2005), and Koppelman & Sethi (2005) considered an additive error term or scale parameter to relax the homogeneity in the variance of the error terms.

There is a possibility to integrate the logit model discourse and weibit model discourse under a closed-form formulation, since both Gumbel and Weibull distributions are in a family of extreme value distributions. A "generalized extreme value distribution" in the field of probability/statistics may play an important role for the integration. The generalized extreme value (GEV) distribution consists of the Gumbel-type, Fréchet-type and Weibull-type extreme value distributions, and has a greater variety in shape than Gumbel or Weibull distribution. The Gumbel-type extreme distribution refers to the Gumbel distribution. Note that the above GEV distribution is different from the GEV distribution that derives the nested logit model and other more elaborate nested logit models (e.g., cross-nested logit model) in the travel behavior analysis. Recently, to avoid confusion, the latter GEV distribution has been referred to as the multivariate extreme value distribution.

Nakayama (2013) proposed a discrete choice model with the GEV-distributed utility. This previous model has a complicated utility function, and it results in unstability of parameter estimation. In this study, we improve the previous model, and propose a more simplified and elaborate formulation of generalized logit model with the GEV-distributed utility. This generalization can avoid one of the limitations of logit model, i.e., the homogeneity of the utility's variance. Furthermore, the generalized logit model includes the (multinomial) weibit model as well as multinomial logit model as special cases, because the GEV distribution combines the Gumbel-type, Fréchet-type and Weibull-type extreme value distributions. Thus, the proposed model unifies the logit and weibit models in a single closed-form expression. On the other hand, the weibit models proposed previously do not include the logit model.

The generalized logit model is incorporated into the transportation network equilibrium model as a route choice model. The network equilibrium model with the generalized logit route choice is formulated as an optimization problem under uncongested networks. The objective function includes Tsallis entropy, which is a type of generalized entropy. Finally, the relationship between Tsallis entropy and generalized logit model with the GEV-distributed utility is examined, and its mathematical framework is elucidated.

2. Extreme value distribution and *q*-exponential function

The logit model has the Gumbel-distributed utility (or error term). The Gumbel distribution is a type of extreme value distribution, and the generalized extreme value (GEV) distribution includes the Gumbel distribution.The cumulative distribution function (CDF) of GEV, $\widetilde{G}(x)$, is expressed as

$$\widetilde{G}(x) = \exp\left\{ -\left[1 + \gamma\left(\frac{x-\mu}{\theta} \right) \right]^{-\frac{1}{\gamma}} \right\} \tag{1}$$

where μ, θ (> 0), and γ are parameters (e.g., Johnson *et al.*, 1995). When $\gamma = 0$, $\widetilde{G}(x) = \exp[-\exp\{-(x-\mu)/\theta\}]$ because $\lim_{\rho\to 0}(1+\rho x)^{1/\rho} = \exp(x)$. This is the CDF of the Gumbel distribution. Thus, the GEV distribution includes the Gumbel distribution as a special case.

Tsallis (1994, 2009) proposed a type of generalization of Boltzmann–Gibbs statistical mechanics and thermodynamics. A core concept in his study is Tsallis entropy, which is the generalization of Boltzman–Gibbs (or Shannon) entropy. Such a generalization is sometimes called "*q*-generalization." The basic operations of the *q*-analysis appear in *q*-generalized statistical mechanics. Tsallis (1994) generalized the exponential function as