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R_{II} type recurrence, generalized eigenvalue problem and orthogonal polynomials on the unit circle

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ABSTRACT

We consider a sequence of polynomials $\{P_n\}_{n\geq 0}$ satisfying a special R_{II} type recurrence relation where the zeros of P_n are simple and lie on the real line. It turns out that the polynomial P_n, for any $n \geq 2$, is the characteristic polynomial of a simple $n \times n$ generalized eigenvalue problem. It is shown that with this R_{II} type recurrence relation one can always associate a positive measure on the unit circle. The orthogonality property satisfied by P_n with respect to this measure is also obtained. Finally, examples are given to justify the results.

1. Introduction

Recurrence relations of the form

$$P_{n+1}(z) = \sigma_{n+1}(z - v_{n+1})P_n(z) - u_{n+1}(z - a_n)(z - b_n)P_{n-1}(z), \quad n \geq 1,$$

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with initial conditions $P_0(z) = 1$ and $P_1(z) = \sigma_1(z - v_1)$, have been studied by Ismail and Masson [14]. In [14] these recurrence relations were referred to as those associated with continued fractions of R_{II} type. One of the interesting results shown in [14] is that given such a recurrence relation, if $u_{n+1} \neq 0$, $P_n(a_k) \neq 0$ and $P_n(b_k) \neq 0$, for all n, k, then associated with this recurrence relation there exists a linear functional $\mathcal{L}$ such that the rational functions $\mathcal{S}_n(z) = P_n(z) \prod_{j=1}^{n} [(z - a_j)^{-1}(z - b_j)^{-1}]$, $n \geq 1$, satisfy the orthogonality

$$\mathcal{L}[z^j \mathcal{S}_n(z)] = 0 \quad \text{for} \quad 0 \leq j < n. \tag{1.1}$$

The importance of these recurrence relations were further highlighted in the work of Zhedanov [23], where the author shows that they are connected to generalized eigenvalue problems involving two tri-diagonal matrices. Just as in [23] we will refer to these recurrence relations as R_{II} type recurrence relations.

Our objective in the present manuscript is to consider the special R_{II} type recurrence relation

$$P_{n+1}(x) = (x - c_{n+1})P_n(x) - d_{n+1}(x^2 + 1)P_{n-1}(x), \quad n \geq 1, \tag{1.2}$$

with $P_0(x) = 1$ and $P_1(x) = x - c_1$, where $\{c_n\}_{n\geq1}$ is a real sequence and $\{d_{n+1}\}_{n\geq1}$ is a positive chain sequence.

By definition $\{d_{n+1}\}_{n\geq1}$ is a positive chain sequence if there exists another sequence $\{g_{n+1}\}_{n\geq0}$ such that $0 \leq g_1 < 1$, $0 < g_{n+1} < 1$ and $(1 - g_n)g_{n+1} = d_{n+1}$ for $n \geq 1$. The sequence $\{g_{n+1}\}_{n\geq0}$ is called a parameter sequence of the positive chain sequence $\{d_{n+1}\}_{n\geq1}$. It is known that a positive chain sequence can have multiple (infinitely many) parameter sequences or just a single parameter sequence. A positive chain sequence always has a minimal parameter sequence.

We denote by $\{\ell_{n+1}\}_{n\geq0}$ the minimal parameter sequence of the positive chain sequence $\{d_{n+1}\}_{n\geq1}$ which is given by $\ell_1 = 0$, $0 < \ell_{n+1} < 1$ and $(1 - \ell_n)\ell_{n+1} = d_{n+1}$ for $n \geq 1$.

By Wall's criteria (see [8, p. 101]) the positive chain sequence $\{d_{n+1}\}_{n\geq1}$ has only a single parameter sequence if and only if the series

$$\sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\ell_{j+1}}{1 - \ell_{j+1}}$$

is divergent. When the positive chain sequence $\{d_{n+1}\}_{n\geq1}$ has multiple parameter sequences one could talk about its maximal parameter sequence $\{M_{n+1}\}_{n\geq0}$ such that

$$0 < g_n < M_n < 1 \quad \text{and} \quad (1 - g_n)g_{n+1} = (1 - M_n)M_{n+1} = d_{n+1}, \quad \text{for} \quad n \geq 1,$$