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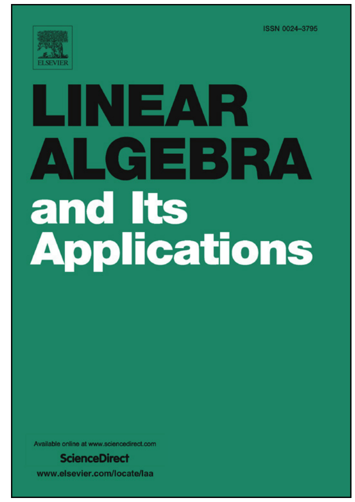

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On Some Sign Patterns of Algebraically Positive Matrices

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Abstract

The concept of an algebraically positive matrix was introduced by Kirkland, Qiao, and Zhan in 2016. A real square matrix A is said to be algebraically positive if there exists a real polynomial f such that $f(A)$ is a positive matrix. In this paper, we give a new characterization of algebraically positive matrices, and characterize all tree sign pattern matrices that allow algebraic positivity, and all star and path sign pattern matrices that require algebraic positivity. Also, all tree sign pattern matrices of order less than 6 requiring algebraic positivity are characterized.

Keywords: Algebraically positive matrix, Tree, Sign pattern, Positive matrix
2010 MSC: 15B35, 15B48, 05C50

1. Introduction

Let $\mathbb{T} = \{+, -, 0, +_0, -_0, \#\}$. Any matrix with entries from $\mathbb{T}$ is said to be a $\mathbb{T}$-pattern matrix. The qualitative class of a $\mathbb{T}$-pattern matrix $\mathcal{A}$ is denoted by $Q(\mathcal{A})$ and is defined by the set of all real matrices obtained from $\mathcal{A}$ replacing $+, -, 0, +_0, -_0, \#$ by a positive, negative, zero, nonnegative, non-positive, and an arbitrary real number respectively. A $\mathbb{T}$-pattern matrix $\mathcal{A}$ is said to allow a property P if at least one matrix in its qualitative class has the property P and $\mathcal{A}$ is said to require a property P if all matrices in its qualitative class have the property P. A sign pattern matrix is a matrix with entries in the set $\{+, -, 0\}$. Let $\mathcal{F}(\mathcal{A})$ denotes the set of all sign pattern matrices obtained from a $\mathbb{T}$-pattern matrix $\mathcal{A}$ by fixing a possible sign in each entry. It is clear that if a $\mathbb{T}$-pattern matrix $\mathcal{A}$ requires a property P, then all sign pattern matrices in $\mathcal{F}(\mathcal{A})$ requires the property P. The negative of a pattern $\mathcal{A}$, denoted by $-\mathcal{A}$, is obtained by multiplying each entry of $\mathcal{A}$ by $-$. A permutation pattern $\mathcal{P}$ is a $\{0, +\}$-pattern, where the symbol $+$ occurs exactly once in each row and each column. A permutation similarity of a pattern $\mathcal{A}$ is a product of the form $\mathcal{P}^T\mathcal{A}\mathcal{P}$, where $\mathcal{P}$ is a permutation pattern. A positive matrix is a matrix all of whose entries are positive real numbers. Most of these notations and terminology are taken from [1, 5] and redefined accordingly to our need.

Classical adjoint of a square matrix A is the transposed matrix of cofactors of A, and it is denoted by $\operatorname{adj}(A)$ (see e.g., [3, p. 22]). Throughout this paper I stands for the identity matrix, $\mathbf{1}$ stands for the column vector with each entry 1 and $\mathbf{0}$ stands for the matrix or the column vector with each entry 0 whose orders will be clear from the context. A^T denotes the transpose of A.

Recall that the graph of a matrix A of order n, denoted by $G(A)$, is defined to be an undirected graph with vertices $1, 2, \ldots, n$ and it has the arc (i, j) if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$. If this graph is a tree, then the matrix A is irreducible if and only if $a_{ij} \neq 0$, whenever $a_{ji} \neq 0$. Analogously the graph of a sign pattern matrix can be defined. An irreducible sign pattern matrix whose graph is a tree is called a tree sign pattern matrix (see e.g, [4]). In particular, if this graph is a star or a path, then the sign pattern matrix is called a star sign pattern matrix or a path sign pattern matrix respectively.

A walk W in $G(A)$ is a sequence $u_0, u_1, u_2, \cdots, u_{t-1}, u_t$ of vertices (need not be distinct) such that u_{p-1}, u_p are adjacent in $G(A)$ for $p = 1, 2, \ldots, t$. If initial and terminal vertices in a walk are same, then the walk is called a circuit (see e.g, [2]). We call the value of the walk W as the product $a_{u_0u_1}a_{u_1u_2}\cdots a_{u_{t-1}u_t}$.

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