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Local automorphisms of finite dimensional simple Lie algebras

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A B S T R A C T

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over an algebraically closed field K of characteristic 0. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is called a local automorphism if for every x in $\mathfrak{g}$ there is an automorphism φ_x of $\mathfrak{g}$ such that $\varphi(x) = \varphi_x(x)$. We prove that a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is local automorphism if and only if it is an automorphism or an anti-automorphism.

1. Introduction

Mappings which are close to automorphisms and derivations of algebras have been extensively investigated: in particular, since the 1990s (see [17], [18], [19]), the description

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of local and 2-local automorphisms (respectively, local and 2-local derivations) of algebras has been deeply studied by many authors.

Given an algebra $\mathcal{A}$ over a field k, a *local automorphism* (respectively, *local derivation*) of $\mathcal{A}$ is a k-linear map $\varphi : \mathcal{A} \to \mathcal{A}$ such that for each $a \in \mathcal{A}$ there exists an automorphism (respectively, a derivation) φ_a of $\mathcal{A}$ such that $\varphi(a) = \varphi_a(a)$. A map $\varphi : \mathcal{A} \to \mathcal{A}$ (not k-linear in general) is called a *2-local automorphism* (respectively, a *2-local derivation*) if for every x, $y \in \mathcal{A}$, there exists an automorphism (respectively, a derivation) $\varphi_{x,y}$ of $\mathcal{A}$ such that $\varphi(x) = \varphi_{x,y}(x)$ and $\varphi(y) = \varphi_{x,y}(y)$.

In [18] the author proves that the automorphisms and the anti-automorphisms of the associative algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices exhaust all its local automorphisms. On the other hand, it is proven in [10] that a certain commutative subalgebra of $M_3(\mathbb{C})$ has a local automorphism which is not an automorphism.

Among other results (see the Introduction of [4] for a detailed historical account), assuming the field k is algebraically closed of characteristic zero, in [1] the authors proved that every 2-local derivation of a finite dimensional semisimple Lie algebra is a derivation; in [2] it is proved that every local derivation of a finite dimensional semisimple Lie algebra is a derivation. As far as automorphisms are concerned, in [9] the authors proved that if $\mathfrak{g}$ is a finite dimensional simple Lie algebra of type A_ℓ ($\ell \geq 1$), D_ℓ ($\ell \geq 4$), or E_i ($i = 6, 7, 8$), then every 2-local automorphism of $\mathfrak{g}$ is an automorphism. This result was extended to any finite dimensional semisimple Lie algebra in [3]. On the other hand, for local automorphisms of simple Lie algebras it is only known that the automorphisms and the anti-automorphisms of the special linear algebra $\mathfrak{sl}(n)$ exhaust all its local automorphisms ([4, Theorem 2.3]).

The main purpose of this paper it to extend this result to any finite dimensional simple Lie algebra: namely we prove that a K-linear endomorphism of a finite dimensional simple Lie algebra $\mathfrak{g}$ over the algebraically closed field K of characteristic zero is a local automorphism if and only if it is an automorphism or an anti-automorphism of $\mathfrak{g}$.

Let G be the connected component of the automorphism group of $\mathfrak{g}$: then G is the adjoint simple algebraic group over K with the same Dynkin diagram as $\mathfrak{g}$. It is clear that every automorphism of $\mathfrak{g}$ is a local automorphism: we show that every anti-automorphism of $\mathfrak{g}$ is a local automorphism too. For this purpose we make use of the Bala–Carter theory for the classification of nilpotent elements in $\mathfrak{g}$.

To show that a local automorphism of $\mathfrak{g}$ is an automorphisms or an anti-automorphisms, we make use of the Tits' Building $\Delta(G)$ of G (as defined in [21, Chap. 5.3]) and the classification theorem [21, Theorem 5.8] which in particular describes the automorphisms of $\Delta(G)$.

2. Preliminaries

Throughout the paper K is an algebraically closed field of characteristic zero. We denote by $\mathbb{R}$ the reals, by $\mathbb{Z}$ the integers.