European Journal of Combinatorics 76 (2019) 88–103

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Triangle-free planar graphs with small independence number

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ARTICLE INFO

Article history:
Received 9 February 2017
Accepted 24 September 2018
Available online xxxx

ABSTRACT

Since planar triangle-free graphs are 3-colourable, such a graph with n vertices has an independent set of size at least $n/3$. We prove that unless the graph contains a certain obstruction, its independence number is at least $n/(3-\varepsilon)$ for some fixed $\varepsilon > 0$. We also provide a reduction rule for this obstruction, which enables us to transform any plane triangle-free graph G into a plane triangle-free graph G' such that $\alpha(G') - |G'|/3 = \alpha(G) - |G|/3$ and $|G'| \leq (\alpha(G) - |G|/3)/\varepsilon$. We derive a number of algorithmic consequences as well as a structural description of n-vertex plane triangle-free graphs whose independence number is close to $n/3$.
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What is the smallest independence number a planar graph on n vertices can have? By Four Colour Theorem, each such graph is 4-colourable, and the largest colour class gives an independent set of size at least $n/4$. On the other hand, there are infinitely many planar graphs for that this bound is tight. In fact, it is an intriguing open problem to describe such graphs, and we do not even know any polynomial-time algorithm to decide whether an n-vertex planar graph has an independent set larger than $n/4$.

In this paper, we study an easier related problem regarding independent sets in planar triangle-free graphs. By Grötzsch' theorem [15], these graphs are 3-colourable, and thus such a graph with n vertices has an independent set of size at least $n/3$. Unlike the general case, this bound is not tight—Steinberg and Tovey [19] proved the lower bound $(n+1)/3$, and gave an infinite family of graphs showing that this bound is tight. Dvořák et al. [10] improved this bound to $(n+2)/3$ except for the graphs from this family. Furthermore, Dvořák and Mnich [11] proved that there exists $\varepsilon > 0$ such that each n-vertex plane triangle-free graph in which every 4-cycle bounds a face has an independent set of size at least $n/(3-\varepsilon)$.

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https://doi.org/10.1016/j.ejc.2018.09.008
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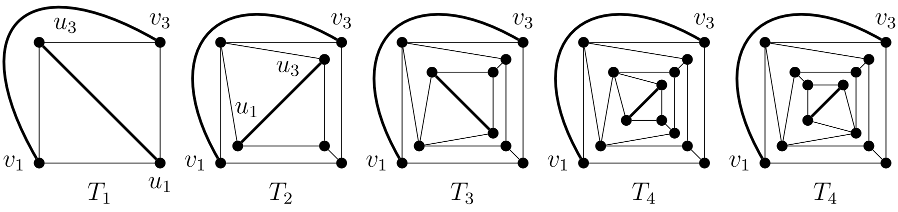

Fig. 1. Some Thomas–Walls graphs (the last two pictures demonstrate two combinatorially distinct drawings of T_4).

Let us recall that $\alpha(G)$ denotes the size of a largest independent set in G. Dvořák and Mnich [11] moreover described an algorithm with time complexity $2^{O(\sqrt{a})}n$ that for an n-vertex planar triangle-free graph G and an integer $a \geq 0$ decides whether $\alpha(G) \geq (n+a)/3$. This algorithm is based on a generalization of the final result of the previous paragraph, which we will state after introducing a couple of definitions.

Let G be a plane graph and let H be a subgraph of G, whose drawing in the plane is inherited from G. Note that each vertex or edge of G that is not contained in H is drawn in some face of H; we say that the faces of H in that some part of G is drawn are *full*. Equivalently, a face f of H is full if and only if f is not a face of G. The supergraph of H that defines the fullness of a face of H will always be clear from the context. By $|G|$, we mean the number of vertices of G. A *k-face* of a plane graph is a face homeomorphic to an open disk bounded by a cycle of length k. We are now ready to state the result.

Theorem 1 (*Dvořák and Mnich [11]*). *There exists a constant $\gamma > 0$ as follows. Let G be a plane triangle-free graph and let H be its subgraph. If every 4-cycle in H bounds a face and every full face of H is a 4-face, then $\alpha(G) \geq \frac{|G|+\gamma|H|}{3}$.*

Hence, if $\alpha(G) \leq (n+a)/3$, then G contains no such subgraph H with more than a/γ vertices, and consequently the tree-width of G is at most $O(\sqrt{a})$; see [11, Lemma 6]. This gives the aforementioned algorithm by using the standard dynamic programming approach to deal with the bounded tree-width graph.

Our main result is a more precise characterization of n-vertex plane triangle-free graphs with no independent set larger than $(n+a)/3$. To state the result, we need to give a few more definitions. We construct a sequence of graphs $T_1, T_2, \ldots$, which we call *Thomas–Walls graphs* (Thomas and Walls [20] proved that they are exactly the 4-critical graphs that can be drawn in the Klein bottle without contractible cycles of length at most 4). Let T_1 be equal to K_4. For $k \geq 1$, let u_1u_3 be any edge of T_k that belongs to two triangles and let T_{k+1} be obtained from $T_k - u_1u_3$ by adding vertices x, y and z and edges u_1x, u_3y, u_3z, xy, xz, and yz. The first few graphs of this sequence are drawn in Fig. 1. Note that although there is a freedom in the construction of T_{k+1} in choosing the edge u_1u_3 and deciding which of its ends is u_1 and which is u_3, all these choices lead to isomorphic graphs (however, from T_4 on, these graphs have several combinatorially distinct drawings in the plane, as is also illustrated in Fig. 1).

For $k \geq 2$, note that T_k contains unique 4-cycles $C_1 = u_1u_2u_3u_4$ and $C_2 = v_1v_2v_3v_4$ such that $u_1u_3, v_1v_3 \in E(G)$. Let $T'_k = T_k - \{u_1u_3, v_1v_3\}$. We also define T'_1 to be a 4-cycle $C_1 = C_2 = u_1v_1u_3v_3$. We call the graphs $T'_1, T'_2, \ldots$ *reduced Thomas–Walls graphs*, and we say that u_1u_3 and v_1v_3 are their *interface pairs*. Let us remark that the tight graphs found by Steinberg and Tovey [19] are precisely those obtained from reduced Thomas–Walls graphs by joining vertices of each interface pair by a path with three edges (and two new vertices).

Let G be a plane triangle-free graph and let H be a subgraph of G isomorphic to a reduced Thomas–Walls graph T'_k. We say that H is a *clean Thomas–Walls k-tube* in G if the 5-faces of H are not full. Our characterization is based on the following strengthening of Theorem 1.

Theorem 2. *For every integer k, there exists a constant $\gamma_k > 0$ as follows. Every plane triangle-free graph without a clean Thomas–Walls k-tube satisfies $\alpha(G) \geq \frac{(1+\gamma_k)|G|}{3}$.*

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