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On the motion of a heavy bead sliding on a rotating wire – Fractional treatment

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ABSTRACT

In this work, we consider the motion of a heavy particle sliding on a rotating wire. The first step carried for this model is writing the classical and fractional Lagrangian. Secondly, the fractional Hamilton's equations (FHEs) of motion of the system is derived. The fractional equations are formulated in the sense of Caputo. Thirdly, numerical simulations of the FHEs within the fractional operators are presented and discussed for some fractional derivative orders. Numerical results are based on a discretization scheme using the Euler convolution quadrature rule for the discretization of the convolution integral. Finally, simulation results verify that, taking into account the fractional calculus provides more flexible models demonstrating new aspects of the real world phenomena.

Introduction

In classical mechanics texts, we faced many problems in which a particle is sliding on a surface or either rotating around an axis. To study and analyze these problems we used the energy dependent method (i.e., Lagrangian and Hamiltonian method). In texts one can find many problems of this kind solved classically using the classical Lagrangian and the classical Hamiltonian.

Nowadays fractional calculus has become an important tools in many areas of science, applied mathematics, physical systems and engineering [1–12], starting from the early development of the generalized Hamiltonian formulation carried by Riewe [13,14]. Riewe has studied non-conservative Lagrangian and Hamiltonian mechanics within fractional calculus. He has used the fractional calculus to obtain a formalism which can be applied for both conservative non-conservative systems. One can obtain the Lagrangian and the Hamiltonian equations of motion for the non-conservative systems. Besides, the generalization of Lagrangian and Hamiltonian fractional mechanics with fractional derivatives were extended and discussed in details [15–21].

The outlines this paper is as follows. In Section "*Basic definitions*" some preliminaries concerning fractional derivatives are presented. In Section "*Classical and fractional description of the physical model*", classical and fractional study has been carried out for the system of interests (a heavy bead sliding on a rotating wire). Section "*Numerical solution and simulation results*" provides numerical solutions of the derived fractional Euler-Lagrange equation for different values of fractional order and initial conditions. Finally, we close the paper by a conclusion in Section "*Primary concepts*".

Basic definitions

In this section, we give in brief some preliminaries concerning the fractional derivatives. There are some definitions for the fractional derivatives including Riemann-Liouville, Weyl, Caputo, Marchaud and Riesz [22,23].

Below we introduce the basic definition of the left and right Caputo fractional derivatives and the left and right Riemann-Liouville fractional integrals. According to [23], let x: $[a, b] \to R$ be a time dependent function. Then, the left and right Caputo fractional derivatives are defined as

$$ {}_a^C D_t^{\alpha} x \triangleq \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(\xi)}{(t-\xi)^{1+\alpha-n}} d\xi, \tag{1} $$

and

$$ {}_t^C D_b^{\alpha} x \triangleq \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{(-1)^n x^{(n)}(\xi)}{(\xi-t)^{1+\alpha-n}} d\xi, \tag{2} $$

respectively, also the left and right Riemann-Liouville fractional integrals are defined as

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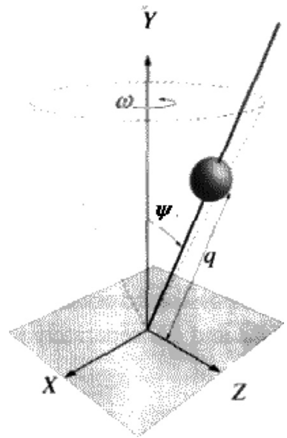

Fig. 1. Heavy bead sliding on a rotating wire.

$$ {}_{t}^{C}I_{b}^{\alpha}x \triangleq \frac{1}{\Gamma(\alpha)}\int_a^t \frac{x(\xi)}{(t-\xi)^{1-\alpha}}d\xi, \tag{3}$$

and

$$ {}_{t}^{RL}I_{b}^{\alpha}x \triangleq \frac{1}{\Gamma(\alpha)}\int_t^b \frac{x(\xi)}{(\xi-t)^{1-\alpha}}d\xi, \tag{4}$$

respectively, where $\Gamma(\cdot)$ denotes the Euler's Gamma function and α represents the fractional derivative and integral order such that $n-1 < \alpha < n$.

Classical and fractional description of the physical model

Our physical system is taken from a famous text in analytical mechanics [24]. In this system we consider a heavy bead sliding without friction on a thin wire that rotated about a vertical axis by a motor at a constant angular frequency w as shown in Fig. 1 below. The wire is tilted away from the y axis by a fixed angle ψ. The bead is constrained to move on the wire, and to describe its motion we need just one dynamical variable which is q (i.e., the distance from origin).

The importance of this example comes from the fact that the kinetic energy in this case depends on both the dynamical variable and on its derivative, instead of on the time derivative alone.

The kinetic energy and the potential energy of the bead respectively read:

$$ T = \frac{1}{2}m(\dot{q}^2 + q^2w^2Sin^2\psi). \tag{5}$$

$$ V = mgqCos\psi. \tag{6}$$

Therefore the classical Lagrangian for our system takes the form

$$ L = T-V = \frac{1}{2}m(\dot{q}^2 + q^2w^2Sin^2\psi)-mgqCos\psi. \tag{7}$$

The classical Euler-Lagrange equation can be obtained as follows

$$ \frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{8}$$

In view of Eqs. (7) and (8), the classical Euler-Lagrange equation reads:

$$ \ddot{q} = qw^2Sin^2\psi-gCos\psi. \tag{9}$$

Now, we can generalize Eq. (7) and write it in fractional form using Caputo fractional derivative as:

$$ L^F = \frac{1}{2}m[({}_a^C D_t^\alpha q)^2 + q^2w^2Sin^2\psi]-mgqCos\psi. \tag{10}$$

The fractional Euler-Lagrange equation (FELE) of motion can also be obtained using the following equation

$$ \frac{\partial L^F}{\partial q} + {}_a^C D_t^\alpha \frac{\partial L^F}{\partial {}_a^C D_t^\alpha q} + {}_a^C D_t^\beta \frac{\partial L^F}{\partial {}_t^C D_b^\alpha q} = 0. \tag{11}$$

As a result of using Eqs. (10) and (11) the FELE reads

$$ {}_t^C D_b^\alpha {}_a^C D_t^\alpha q = gCos\psi-qw^2Sin^2\psi. \tag{12}$$

As $\alpha \to 1$, the FELE (12) reduces to the classical Euler-Lagrange equation (9).

In a recent work Bourdin [25] derive the conditions that ensure the existence of a weak solution for fractional Euler-Lagrange equations similar to those obtained in our system.

In the following, we are going to obtain the fractional Hamilton's equation (FHEs) of motion. For this purpose, we have to introduce the following two generalized momenta $P_{\alpha,q}$, and $P_{\beta,q}$

$$ P_{\alpha,q} = \frac{\partial L^F}{\partial {}_a^C D_t^\alpha q} = m\,{}_a^C D_t^\alpha q. \tag{13}$$

$$ P_{\beta,q} = \frac{\partial L^F}{\partial {}_t^C D_b^\beta q} = 0. \tag{14}$$

As a result, the fractional Hamiltonian function can be obtained from

$$ H^F = P_{\alpha,q}\,{}_a^C D_t^\alpha q + P_{\beta,q}\,{}_t^C D_b^\beta q-L^F. \tag{15}$$

Substituting Eqs. (13) and (14) into Eq. (15), we obtained

$$ H^F = \frac{1}{2}m[({}_a^C D_t^\alpha q)^2-q^2w^2Sin^2\psi] + mgqCos\psi. \tag{16}$$

Now, the FHEs of motion are obtained as

$$ \frac{\partial H^F}{\partial q} = {}_t^C D_b^\alpha P_{\alpha,q} + {}_a^C D_t^\beta P_{\beta,q} \Rightarrow {}_t^C D_b^\alpha {}_a^C D_t^\alpha q = gCos\psi-qw^2Sin^2\psi. \tag{17}$$

Notice that the above FHEs of motion (17) is the same as the corresponding FELEs in Eqs. (12). Again as $\alpha \to 1$ Eq. (17) reduces to the CELEs (9). Our aim now is to obtain the numerical solutions of Eq. (17) for different values of α.

Numerical solution and simulation results

In this section, we propose the approximate analytical solution for the problem (17) with $0 < \alpha \leqslant 1$. We consider the initial conditions as follows

$$ q(a) = q_0\,, \tag{18}$$

$$ q'(a) = q_1. \tag{19}$$

Recently, we used Bernstein operational matrices of Caputo derivative, Riemann-Liouville fractional integral and product for solving fractional quadratic Riccati differential equations [26], multi-order fractional differential equations [27,28], nonlinear system of fractional differential equations [29] and multi-dimensional fractional optimal control problems with inequality constraint [30,31]. Now, we apply the Bernstein operational matrices method for solving the problem (17)–(19).

Primary concepts

Referring to Ref. [32], we can see details of proposed concepts in here.

Le $\beta_{i,m}(t) = \binom{m}{i}\frac{(t-a)^i(b-t)^{m-i}}{(b-a)^m}$, $i = 0, 1, \cdots, m$ are the Bernstein polynomials of degree m on interval $[a, b]$ and $\Psi_m(t) = [\beta_{0,m}(t), \beta_{1,m}(t), ..., \beta_{m,m}(t)]^T$. Since the Bernstein polynomials form a basis on $[a, b]$, for any function $y \in C^{m+1}[a, b]$ we can approximate $y(t)$ as follows

$$ y(t) \approx \sum_{i=0}^{m} c_i\beta_{i,m}(t) = c^T\Psi_m(t), \tag{20}$$