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Research Paper

A smooth hyperbolic approximation to the Generalised Classical yield function, including a true inner rounding of the Mohr-Coulomb deviatoric section

Alexander M. Lester*, Scott W. Sloan

Centre of Excellence for Geotechnical Science and Engineering, The University of Newcastle, Callaghan NSW 2308, Australia

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A B S T R A C T

A new yield function recently introduced by Lagioia and Panteghini (2016), herein referred to as the Generalised Classical (GC) yield function, combines a series of criteria commonly used in geotechnical analysis into a single equation, including those of Tresca, Mohr-Coulomb and Matsuoka-Nakai. This makes for efficient implementation of multiple criteria into finite element software, and in this paper two key improvements are made to further enhance the usefulness of the GC yield function. The first is the development of a new expression for the shape parameter γ, corresponding to the so-called 'Inner Mohr-Coulomb' option, which ensures that a true inner rounding of the hexagonal Mohr-Coulomb deviatoric section is always obtained. The second is the introduction of a hyperbolic rounding to eliminate a discontinuity which can occur at the tip in the meridional section of the GC yield surface. The resulting yield surface is at least C^2 continuous everywhere, provided a rounded criterion is selected, and can thus be used in consistent tangent finite element formulations. The results of finite element analyses carried out for two benchmark problems (a thick cylinder and a rigid strip footing) demonstrate the benefits of the rounding techniques in the new yield surface. Comparisons are made with the original yield surface and also the Hyperbolic Rounded Mohr-Coulomb (HRMC) yield surface originally developed by Abbo and Sloan (1995).

1. Introduction

A key component of an elastoplastic constitutive model of material behaviour is the yield function, which defines the onset of plastic deformation for all possible stress paths imposed upon the material. When implemented into finite element software it is preferable, particularly with adaptive explicit stress integration schemes, that the yield function be defined as a continuous and differentiable surface in 3D principal stress space, such that its gradients can be evaluated at any point on the surface. Often, many different yield functions will be implemented separately and various rounding techniques will be used to eliminate any discontinuities that arise. Recently, a new form of yield function (herein referred to as the 'Generalised Classical' or GC yield function) has been proposed by Lagioia and Panteghini [4]. This combines many of the so-called 'classical' yield criteria, namely those of von Mises, Drucker-Prager, Tresca, Mohr-Coulomb, Matsuoka-Nakai and Lade-Duncan, in a single equation. This means that, instead of having to define multiple criteria separately, the generalised equation can be implemented and used for finite element analyses with any one of these criteria.

In this paper, two key improvements to the GC yield function are introduced. The first is a new expression for the shape parameter γ for the Inner Mohr-Coulomb option available within the yield function. The use of this expression ensures that a true inner rounding of the hexagonal deviatoric section of the Mohr-Coulomb yield surface is obtained, which lies inside the unrounded Mohr-Coulomb hexagon everywhere. The second improvement is a hyperbolic rounding of the tip discontinuity which occurs at the junction between meridional sections of the yield surface when using a frictional criterion, such as Mohr-Coulomb. This ensures that there are no discontinuities left in the yield surface. Together, the two improvements define a true inner rounding of the Mohr-Coulomb yield surface, which is at least C^2 continuous everywhere.

The GC yield function is discussed with reference to the form that it takes as a yield surface plotted in principal stress space. Yield surfaces are often described using their form in the deviatoric plane, whose unit normal points along the diagonal of principal stress space, and the meridional plane, whose unit normal is perpendicular to the space diagonal. Additionally, instead of being defined in terms of the principal stresses $(\sigma_1,\sigma_2,\sigma_3)$ directly, the yield surface is often expressed using a set of stress invariants (p,J,θ) which are based on a separation of

* Corresponding author.
E-mail address: alexander.lester@uon.edu.au (A.M. Lester).

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stresses into the meridional and deviatoric planes. These invariants are written in terms of the usual Cartesian stress components $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$ as:

$$p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \tag{1}$$

$$J = \sqrt{\frac{1}{2}(s_x^2 + s_y^2 + s_z^2) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2} \tag{2}$$

$$\theta = \frac{1}{3}\sin^{-1}\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J^3}\right) \tag{3}$$

where

$$J_3 = s_x s_y s_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - s_x\tau_{yz}^2 - s_y\tau_{xz}^2 - s_z\tau_{xy}^2$$

and

$$s_x = \sigma_x - p$$

$$s_y = \sigma_y - p$$

$$s_z = \sigma_z - p$$

The GC yield surface is now introduced as a function of the stress invariants (p, J, θ). Adopting the convention that compressive stresses are positive, the yield surface may be expressed as:

$$f = F(p) + J\Gamma(\theta) = 0 \tag{4}$$

where

- $F(p) = -(Mp + K)$ defines the meridional section.
- $\Gamma(\theta) = \alpha\cos\left[\frac{1}{3}\cos^{-1}(\beta\sin 3\theta) - \frac{\pi}{6}\gamma\right]$ defines the deviatoric section.
- $M = \frac{6\sin\phi}{\sqrt{3}(3-\sin\phi)}$ is the magnitude of the slope of the meridional section under triaxial compression conditions.
- $K = cM\cot\phi = \frac{6c\cos\phi}{\sqrt{3}(3-\sin\phi)}$ is the J intercept of the meridional section under triaxial compression conditions. This was mistakenly written as $K = c\cot\phi$ in [4].
- α, β, γ are shape parameters whose values depend upon the chosen classical criterion (von Mises, Drucker-Prager etc.) as well as the value of ϕ. A complete listing of values is provided in Table 3 of [4], and these are reproduced in Appendix A with the exception of γ for the Inner Mohr-Coulomb option, where the newly-derived expression is now included (see Section 2).
- c and ϕ are the cohesion and friction angle respectively, which are the usual strength parameters for a material modelled using one of the classical yield criteria.

In addition to its ability to combine multiple, widely used yield criteria under the one equation, the GC yield surface carries a number of other features which render it superior to other yield surfaces:

- The shape parameter β, in addition to altering the shape of the deviatoric section to suit various yield surfaces, also acts as a rounding parameter for the Tresca and Mohr-Coulomb criteria which have corner discontinuities in their deviatoric sections. By choosing a value of β slightly less than 1, a Tresca or Mohr-Coulomb yield surface with a rounded deviatoric section can be recovered, which is especially useful for finite element applications.
- Aside from the tip discontinuity which occurs in the meridional plane when using frictional criteria such as the Drucker-Prager or Mohr-Coulomb models, the yield surface is at least C^2 continuous everywhere when $\beta \neq 1$. This makes it suitable for use in finite element formulations which adopt either explicit or implicit stress integration schemes.
- The yield surface is convex everywhere for all criteria except the Outer Mohr-Coulomb variant, where it is convex for most practical choices of parameters.
- The meridional and deviatoric sections of the yield surface are mathematically independent of each other, which makes it very easy to alter their form as desired.

In this paper, a new expression will be derived for the shape parameter γ which corresponds to the Inner Mohr-Coulomb option of the GC yield function. This new expression will result in a true inner rounding of the Mohr-Coulomb hexagon in the deviatoric plane. A hyperbolic approximation to the GC yield surface, known as the Hyperbolic Generalised Classical (HGC) yield surface, will then be formulated. The gradients and gradient derivatives to the GC and HGC yield surfaces, which are necessary for their implementation in finite element codes, will then be derived. Two subroutines (YIELD and GRAD), written in the Fortran 77 programming language will then be introduced, to illustrate how the yield surfaces may be incorporated efficiently in a finite element package. These subroutines have been implemented into SNAC, a finite element program developed at the University of Newcastle, Australia, and analyses have been carried out for two benchmark problems (a thick cylinder and a rigid strip footing), the results of which are presented herein.

2. New expression for γ corresponding to the Inner Mohr-Coulomb criterion

Lagioia and Panteghini [4] noted that the Inner Mohr-Coulomb option of the GC yield function does not strictly inscribe the hexagonal deviatoric section of the unrounded Mohr-Coulomb criterion, and demonstrated this by way of their own finite element analyses. The extent to which the rounded deviatoric section passes outside the unrounded section is not significant if the value of β is very close to 1, and Lagioia and Panteghini [4] suggested that choosing $\beta = 0.9999$ leads to a very close approximation of the unrounded section. However, as the results of finite element analyses in Section 7 will show, even small reductions in the value of β can place the rounded section a significant distance outside the unrounded section, leading to unconservative estimates of collapse loads in practical problems.

It can be shown that, by deriving an alternative expression for the shape parameter γ, a true inner rounding of the Mohr-Coulomb deviatoric section can be achieved. The original expression for γ used in [4] for the Inner Mohr-Coulomb option is:

$$\gamma = 1 - \bar{\gamma} \tag{5}$$

where

$$\bar{\gamma} = \frac{6}{\pi}\tan^{-1}\left(\frac{\sin\phi}{\sqrt{3}}\right)$$

This expression for γ works when using the unrounded Mohr-Coulomb option (i.e. $\beta = 1$), and indeed should be recoverable from any proposed new expression for γ where the Inner Mohr-Coulomb option is selected. Following the derivation outlined in Appendix B, an appropriate expression for γ corresponding to the Inner Mohr-Coulomb option is found to be:

$$\gamma = \frac{2}{\pi}\cos^{-1}\left[\beta\sin\left(\frac{\pi}{2}\bar{\gamma}\right)\right] \tag{6}$$

If β is set to 1 then Eq. (6) reduces to Eq. (5) as required, and so Eq. (6) may be used to furnish a true inner rounding of the Mohr-Coulomb deviatoric section, where β is set to a value less than 1.

3. Formulation of hyperbolic yield surface

A 'tip' discontinuity between the meridional sections of the GC yield surface exists when a frictional criterion such as Mohr-Coulomb is used, and this scenario is depicted in Fig. 1. Whilst any 'corner' discontinuities in the deviatoric plane can be eliminated by choosing an appropriate value of the shape parameter β, the yield surface as written in Eq. (4) does not allow for any rounding in the meridional plane. This