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Topologies of strategically formed social networks based on a generic value function—Allocation rule model

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ABSTRACT

A wide variety of game theoretic models have been proposed in the literature to explain social network formation. Topologies of networks formed under these models have been investigated, keeping in view two key properties, namely efficiency and stability. Our objective in this paper is to investigate the topologies of networks formed with a more generic model of network formation. Our model is based on a well known model, the value function – allocation rule model. We choose a specific value function and a generic allocation rule and derive several interesting topological results in the network formation context. A unique feature of our model is that it simultaneously captures several key determinants of network formation such as (i) benefits from immediate neighbors through links, (ii) costs of maintaining the links, (iii) benefits from non-neighboring nodes and decay of these benefits with distance, and (iv) intermediary benefits that arise from multi-step paths. Based on this versatile model of network formation, our study explores the structure of the networks that satisfy one or both of the properties, efficiency and pairwise stability. The following are our specific results: (1) we first show that the complete graph and the star graph are the only topologies possible for non-empty efficient networks; this result is independent of the allocation rule and corroborates the findings of more specific models in the literature. (2) We then derive the structure of pairwise stable networks and come up with topologies that are richer than what have been derived for extant models in the literature. (3) Next, under the proposed model, we state and prove a necessary and sufficient condition for any efficient network to be pairwise stable. (4) Finally, we study topologies of pairwise stable networks in some specific settings, leading to unravelling of more specific topological possibilities.

1. Introduction

Examples of networks in the field that form as a result of autonomous agents seeking to fulfill their individual objectives include the Internet, peer-to-peer networks, and social networks. In this paper, we focus on social networks which are social structures comprised of individuals connected by one or more relationships (Wasserman and Faust, 1994). In this paper, graphs represent social networks with nodes indicating individuals and edges indicating the social interactions connecting them.

It is well known that social networks play an important role in spreading ideas and information (Boorman, 1975; Schelling, 1978; Cooper, 1982; Rogers, 1995; Valente, 1995; Strang and Soule, 1998; Calvo-Armengol, 2004; Calvo-Armengol and Jackson, 2004; Jackson and Yariv, 2005, 2006; Jackson and Rogers, 2007; Leskovec et al., 2007). Individuals that disseminate information in social networks receive benefits and incur costs in terms of money, time, and effort as a consequence of the links with other individuals. As individuals incur costs, they act strategically while selecting their immediate neighbors. It is important to understand the effect of the strategic behavior of the individuals on the formation of social networks. Recently, many researchers have proposed several models of social network formation using game theoretic approaches (Jackson and Wolinsky, 1996; Dutta et al., 1998; Johnson and Gilles, 2000; Dutta and Jackson, 2000; Slikker and van den Nouweland, 2001; Jackson and Watts, 2002; Jackson and van den Nouweland, 2005; Jackson, 2003, 2005, 2008; Demange and Wooders, 2005; Goyal, 2007; Hummon, 2000; Galeotti et al., 2006; Doreian, 2006; Buskens and van de Rijt, 2008; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008; Doreian, 2008a,b). The crux of most of these studies is the underlying strategic form game (Myerson (1997)) where the players, strategies, and utilities are defined as follows: (i) the individuals are the players, (ii) the choice of neighbors is the strategy of each individual, and (iii) the utility of each individual depends on its neighborhood and the structure of the network. The main emphasis

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of these models (Jackson and Wolinsky, 1996; Johnson and Gilles, 2000; Goyal, 2007; Jackson, 2008; Hummon, 2000; Galeotti et al., 2006; Doreian, 2006; Buskens and van de Rijt, 2008; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008; Doreian, 2008a,b) is to study the stability and efficiency properties of the networks that emerge. Informally, a network is said to be stable if it is in some strategic equilibrium[1] (Myerson, 1977) and we call a network efficient if the sum of the utilities of the nodes in the network is maximal. Some of the studies (Jackson and Wolinsky, 1996; Johnson and Gilles, 2000; Slikker and van den Nouweland, 2001; Goyal, 2007; Hummon, 2000; Galeotti et al., 2006; Doreian, 2006; Buskens and van de Rijt, 2008; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008; Doreian, 2008a,b) yield precise predictions on the network topologies that result, if stability and efficiency are to be satisfied.

Our objective in this paper is to investigate the structure or topologies of social networks formed using a model of network formation. Our model is based on a well known model, the value function – allocation rule model that has been proposed by Jackson and Wolinsky (1996), which is investigated later, among others, by Johnson and Gilles (2000), Dutta and Jackson (2000), Slikker and van den Nouweland (2001), Watts (2001), Jackson and Watts (2002), Jackson (2003, 2005), Galeotti et al. (2006), Bloch and Jackson (2007), and Goyal (2007). We choose a specific value function that captures several important aspects of network formation and we choose a class of allocation rules satisfying a set of appropriate axioms. We work with a rich class of allocation rules rather than a single specific allocation rule, and in this sense, the proposed model becomes a generic model. Using this model, we derive several interesting topological predictions in the network formation context.

The game theoretic model that we work with in this paper is a strategic form game where individuals announce independently the links they wish to form to other individuals and the links are formed under mutual consent. For each graph that emerges due to the strategies of the individuals, we define a network value of the graph using a value function. This value function satisfies certain desirable properties. The network value is divided among the nodes as utilities of the nodes, using any allocation rule that satisfies a set of appropriate axioms. The class of allocation rules considered includes well known allocation rules such as the Myerson value[2] (Myerson, 1977; Hart and Kurz, 1983; Owen, 1986; Moulin, 1988; Jackson and Wolinsky, 1996; Dutta et al., 1998; Chwe, 2000; Slikker and van den Nouweland, 2001; Kar, 2002; Jackson and Watts, 2002; Faigle and Kern, 1992; Goyal and Joshi, 2003) and makes the network formation model a generic one. The combination of the value function and the allocation rule ensures that the utilities of the nodes are decided by key determinants of network formation such as:

1. *Link Benefits*: Benefits that individual nodes derive from immediate neighbors through direct links (Jackson and Wolinsky, 1996; Slikker and van den Nouweland, 2001; Demange and Wooders, 2005; Goyal, 2007; Jackson, 2008; Hummon, 2000; Doreian, 2006; Buskens and van de Rijt, 2008; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008).
2. *Link Costs*: Costs to individual nodes to maintain the above links (Jackson and Wolinsky, 1996; Slikker and van den Nouweland, 2001; Demange and Wooders, 2005; Goyal, 2007; Jackson, 2008; Hummon, 2000; Doreian, 2006; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008).
3. *Benefits from Non-neighbor Nodes*: It is well understood in social networks that individuals gain more advantages from their immediate neighbors compared to that from the neighbors of these neighbors and so on (Jackson and Wolinsky, 1996; Slikker and van den Nouweland, 2001; Demange and Wooders, 2005; Goyal, 2007; Jackson, 2008; Hummon, 2000; Doreian, 2006; Buskens and van de Rijt, 2008; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008; Johnson and Gilles, 2000; Bloch and Jackson, 2007; Calvo-Armengol, 2004; Dutta et al., 1998; Dutta and Jackson, 2000; Galeotti et al., 2006). We model the benefits from non-neighbor nodes through a benefit function that captures the decay of benefits with the (shortest) distance between the source node and the target node.
4. *Intermediary Benefits*: When information flows from one individual node to another in a network using multi-step paths, the individual nodes on the multi-step paths can/do gain appropriate intermediary benefits. These intermediary benefits are known as bridging benefits and the theory of structural holes describes this phenomenon in detail (Burt, 2001, 2004, 2002, 2005, 2007, 1992; Goyal, 2007; Jackson, 2008). Such situations are prevalent in many practical situations such as (i) decentralized peer-to-peer file-sharing systems, for example Gnutella (Kleinberg and Raghavan, 2005; Lua et al., 2005), (ii) job finding through social networks that provide indirect access to a large set of people connected through multi-step paths of acquaintances (Jackson, 2008), and (iii) query incentive networks (Kleinberg and Raghavan, 2005). In particular, the notion of structural holes is effectively captured in a few models of social network formation in the literature (Goyal and Vega-Redondo, 2007; Buskens and van de Rijt, 2008; Kleinberg et al., 2008).

In this network formation setting, we investigate the properties as well as the topological structure of networks that are efficient and that are pairwise stable (Jackson and Wolinsky, 1996). The notion of efficiency that we consider is maximization of the sum of utilities of the nodes. Efficient networks are important as they are the most productive from the overall network viewpoint. A network is said to be pairwise stable if (i) no node has an incentive to sever a link, and (ii) no pair of nodes has an incentive to form a new link. Pairwise stable networks are important as such networks are robust to the strategic behavior of the nodes. We also investigate the compatibility of pairwise stability and efficiency.

We emphasize that our model is more general than several existing models in the literature in terms of capturing key determinants of network formation and investigating the properties as well as the topological structure of networks that satisfy pairwise stability and efficiency. In the following section, we present the relevant work and bring out the contributions of our work.

1.1. Relevant work

We must note that the field of network formation is very rich (Jackson, 2003, 2005, 2008; Goyal, 2007; Demange and Wooders, 2005; Slikker and van den Nouweland, 2001; Hummon, 2000; Doreian, 2006; Buskens and van de Rijt, 2008; Goyal and Vega-Redondo, 2007; Kleinberg et al., 2008; Johnson and Gilles, 2000; Bloch and Jackson, 2007; Calvo-Armengol, 2004; Dutta et al., 1998; Dutta and Jackson, 2000; Galeotti et al., 2006; Jackson and Wolinsky, 1996; Jackson and Watts, 2002; Jackson and van den Nouweland, 2005; Doreian, 2008a,b). We have only included a discussion of the models that are most relevant to our work. At the end of this section, we highlight the research gap in the literature. When there is no confusion, we use the words *graph* and *network* synonymously.

[1] Various notions of stability notions are present in the literature such as pairwise stability (Jackson and Wolinsky, 1996), Nash stability (Goyal, 2007), unilateral stability (Buskens and van de Rijt, 2008).

[2] Please refer to Appendix A for more details on Myerson value.