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Physics of day-to-day network flow dynamics

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ABSTRACT

This paper offers a new look at the network flow dynamics from the viewpoint of physics by demonstrating that the traffic system, in terms of the aggregate effects of human behaviors, may exhibit like a physical system. Specifically, we look into the day-to-day evolution of network flows that arises from travelers' route choices and their learning behavior on perceived travel costs. We show that the flow dynamics is analogous to a damped oscillatory system. The concepts of energies are introduced, including the potential energy and the kinetic energy. The potential energy, stored in each link, increases with the traffic flow on that link; the kinetic energy, generated by travelers' day-to-day route swapping, is proportional to the square of the path flow changing speed. The potential and kinetic energies are converted to each other throughout the whole flow evolution, and the total system energy keeps decreasing owing to travelers' tendency to stay on their current routes, which is analogous to the damping of a physical system. Finally, the system will approach the equilibrium state with minimum total potential energy and zero kinetic energy. We prove the stability of the day-to-day dynamics and provide numerical experiments to elucidate the interesting findings.

1. Introduction

The notion of user equilibrium (UE), as the norm for transportation system analysis, describes the ideal static state of the transportation networks as a result of the aggregate behavior of road users when they are all rational utility-maximizers. It was first proposed by Wardrop (1952) and then further extended to stochastic user equilibrium (SUE) (Daganzo and Sheffi, 1977), boundedly rational user equilibrium (BRUE) (Mahmassani and Chang, 1987) and so on, attempting to predict traffic flows with more realistic assumptions on travelers' behavior.

The UE traffic assignment problem was formulated as a mathematical programming problem in Beckmann et al. (1956), known as the "Beckmann's transformation". Although the formulation reflects some inherent properties of the traffic network, it is in a long time viewed as no intuitive economic or behavioral interpretation but merely a mathematical construct to solve UE (Sheffi, 1984). In economics, traffic assignment problem with separable link cost functions is modeled as the potential game (Rosenthal, 1973). In a potential game, all players' strategies are mapped into one global potential function. The difference in values for the potential function has the same value as the payoff of each player when changing one's strategy ceteris paribus. The equilibrium is reached at the local optima of the potential function. From this point of

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view, Beckmann's transformation can be endowed with a physical-like meaning - the "potential" of the transportation network. The equilibria are obtained when the system achieves the lowest potential (Monderer and Shapley, 1996; Sandholm, 2001).

Most studies on static traffic assignment examine the final equilibrium state when travelers have no incentive for route swapping. However, in a real traffic network, it is always observed that traffic flows fluctuate from time to time (Guo and Liu, 2011; He and Liu, 2012), due to the interference of external factors and change of the network itself. Furthermore, a disequilibrated transportation network would incline to approach the equilibrium (Guo and Liu, 2011; He and Liu, 2012) through travelers' route swapping. To explain the mechanism of network flow fluctuation and attainment of UE states, a substantial stream of research has been developed to look into the "day-to-day" flow dynamics.

The flow dynamics is sometimes described by a set of deterministic ordinary differential equations (ODEs) (Cho and Hwang, 2005; Han and Du, 2012; He et al., 2010; He and Liu, 2012; Smith and Mounce, 2011). A "rational behavior adjustment process" (RBAP) with fixed demand was proposed by Zhang et al. (2001) and Yang and Zhang (2009), assuming that "the aggregated travel cost in the system based on the previous day's route travel costs will decrease when the route flows change from day to day" (Yang and Zhang, 2009). Many typical models follow RBAP, such as those proposed by Smith (1984), Friesz et al. (1994) and Nagurney and Zhang (1997). An equivalent link-based "discrete rational adjustment process" was proposed by Guo et al. (2013, 2015). The BRUE-based day-to-day dynamics was investigated recently in Di et al. (2015), Guo and Liu (2011), Guo (2013) and Wu et al. (2013).

In another branch of the literature, fluctuation of network flows is examined as a result of travelers' perception and day-to-day learning on route travel costs (Bie and Lo, 2010; Cantarella and Cascetta, 1995; Cascetta and Cantarella, 1993; Horowitz, 1984; Watling, 1999; Xiao and Lo, 2015; Ye and Yang, 2013). Travelers are assumed to possess their own perception on future traffic conditions and choose routes based on their perception. The perceived cost is updated according to new experience or real-time traffic information, and the route choice is modeled as a stochastic network loading process, given the new perceived costs. The corresponding stationary state of these models is SUE.

The above two types of models are based on different assumptions. The first type deals with traffic flow evolution by updating traffic flows based on actual traffic conditions, but it ignores the impact of historical traffic information on travelers' route choice decisions. In contrast, the second type treats the flow swapping as a result of cognition changing from a more intrinsic aspect, but it is more difficult to verify since the cognition is more difficult to measure and estimate than flows.

In this paper, we explore the mechanism of travelers' learning behavior and route swapping behavior in an integrated manner. Road users on the routes with higher costs will tend to switch to the ones with lower costs, while taking history into account. As a result, their route swapping speeds depend on the route travel cost differences both currently and in the past. Such behavior is defined as the "inertia" in day-to-day flow dynamics in this study. When introducing "inertia" into the continuous route swapping model, the day-to-day dynamics can be described by a set of second-order ODEs, which is similar to the physical motion equation of a harmonic oscillator. By analogy to the physical system, we are able to identify the "damping factor", "restoring force", "potential energy" and "kinetic energy" of the network during the day-to-day evolution. The difference between the actual costs on each route-pair acts like "restoring force" in the transportation network, while the route swapping brings "damping". Beckmann's transformation is translated into the "potential energy", as it was treated in the previous literature (Jin, 2007; Peeta and Yang, 2003; Sandholm, 2001), and the "kinetic energy" is defined to be associated with the flow changing. Total energy of the transportation network now comprises both the potential energy and the kinetic energy. The system keeps losing energy due to the "friction" caused by travelers' tendency to stay on their current routes and eventually reaches UE. The UE state is also the minimum potential energy state with zero kinetic energy, which is consistent with the *minimum total potential energy principle* (Hashin and Shtrikman, 1963).

The rest of this paper is organized as follows. Section 2 develops a continuous-time day-to-day dynamical model by considering travelers' learning process and route swapping behavior. The day-to-day flow dynamics is formulated as a set of second-order ODEs, which possesses a similar form to the motion equation of a harmonic oscillator. Section 3 briefly introduces the dynamics and energies of a damped oscillatory system. By analogy to a damped oscillatory system, formulae of kinetic and potential energies of the dynamical traffic network are developed and relationship between network energy and traffic equilibrium is discussed. In Section 4, stability analysis is provided by LaSalle's theorem. The total mechanical energy function is chosen to be the Lyapunov function. Section 5 examines some interesting properties of the day-to-day model by numerical examples. The last section concludes the study and highlights some future research directions.

2. The second-order day-to-day dynamics

Consider a directed traffic network $G = (N, A)$ consisting of a set N of nodes and a set A of links. Let W denote the set of origin-destination (OD) pairs and R_w the set of all paths connecting OD pair $w \in W$. Let $d_w > 0$ be the traffic demand between OD pair $w \in W$, f_{rw} the flow on path $r \in R_w$ between OD pair $w \in W$ and v_a the flow on link $a \in A$. Let $|R_w|$ represent the cardinality of set R_w, i.e., there are $|R_w|$ number of paths connecting OD pair $w \in W$. Each link has a separable cost function $c_a(v_a)$, which is assumed to be nonnegative, differentiable, convex and strictly increasing. Define $\Delta = [\delta_{ar}]$ and $\Lambda = [\lambda_{rw}]$ as the link-path and OD-path incidence matrices, respectively, where δ_{ar} equals 1 if path r uses link a and 0 otherwise, and λ_{rw} equals 1 if path r connects OD pair w and 0 otherwise.

Let $\mathbf{f} = (f_{rw}, r \in R_w, w \in W)^{\mathrm{T}}$, $\mathbf{d} = (d_w, w \in W)^{\mathrm{T}}$ and $\mathbf{v} = (v_a, a \in A)^{\mathrm{T}}$ be the vectors of path flows, OD demands and link flows, respectively, where superscript "T" represents the transpose operation. The relationship between link flows, path flows