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Elastic demand dynamic network user equilibrium: Formulation, existence and computation

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ABSTRACT

This paper is concerned with dynamic user equilibrium with elastic travel demand (E-DUE) when the trip demand matrix is determined endogenously. We present an infinite-dimensional variational inequality (VI) formulation that is equivalent to the conditions defining a continuous-time E-DUE problem. An existence result for this VI is established by applying a fixed-point existence theorem (Browder, 1968) in an extended Hilbert space. We present three computational algorithms based on the aforementioned VI and its re-expression as a *differential variational inequality* (DVI): a projection method, a self-adaptive projection method, and a proximal point method. Rigorous convergence results are provided for these methods, which rely on increasingly relaxed notions of generalized monotonicity, namely *mixed strongly-weakly monotonicity* for the projection method; *pseudomonotonicity* for the self-adaptive projection method, and *quasimonotonicity* for the proximal point method. These three algorithms are tested and their solution quality, convergence, and computational efficiency are compared. Our convergence results, which transcend the transportation applications studied here, apply to a broad family of VIs and DVIs, and are the weakest reported to date.

1. Introductory remarks

This paper is concerned with an extension of the *simultaneous route-and-departure-time* (SRDT) dynamic user equilibrium (DUE) originally proposed in Friesz et al. (1993) and discussed subsequently by Friesz et al. (2001, 2013b, 2011), Friesz and Meimand (2014), and Friesz and Mookherjee (2006). Specifically, the model of interest herein relaxes the assumption of fixed trip volumes by considering elastic travel demands among origin–destination pairs. The extension of DUE based on a fixed trip table to the explicit consideration of elastic demand is not a straightforward matter. In particular, one has to show a dual variable associated with arrival time is equivalent to an adjoint (co-state) variable by exploiting the transversality conditions familiar from differential variational inequality (DVI) theory. The first such analysis was performed by Friesz and Meimand (2014), who employed separable demand functions for each origin–destination pair. They used a variational calculus approach, which, although correct, masks many of the measure-theoretic arguments essential to understanding the generality of a DVI representation of the elastic-demand DUE (E-DUE) problem in continuous time. By contrast, this paper not only considers nonseparable demand functions, but it also provides all measure-theoretic arguments needed to understand the

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DVI formulation. Furthermore, this paper presents an existence theory, three algorithms and their proofs of convergence, and numerical studies, all of which are missing from Friesz and Meimand (2014). That is to say, this paper provides the first complete mathematical and numerical analysis of the SRDT E-DUE problem.

1.1. Dynamic user equilibrium with elastic demand: some review

Most of the studies of DUE reported in the *dynamic traffic assignment* (DTA) literature are about dynamic user equilibrium with constant travel demand for each origin–destination pair. It is, of course, not generally true that travel demand is fixed, even for short time horizons. Arnott et al. (1993) and Yang and Huang (1997) directly consider elastic travel demand in the context of a single bottleneck. Yang and Meng (1998) extend a simple bottleneck model to a general queuing network with known elastic demand functions for each origin–destination (O–D) pair. They employ a *space–time expanded network* (STEN) representation of the network loading submodel. Wie et al. (2002) study a version of the dynamic user equilibrium with elastic demand, using a complementarity formulation that requires path delays to be expressible in closed form. Szeto and Lo (2004) study dynamic user equilibrium with elastic travel demand when network loading is based on the *cell transmission model* (CTM); their formulation is discrete-time in nature and is expressed as a finite-dimensional variational inequality (VI). The VI is solved with a descent method under the assumption that the delay operator is co-coercive. Han et al. (2011) study dynamic user equilibrium with elastic travel demand for a network with a single origin–destination pair whose traffic flow dynamics are also described by the CTM; the CTM is chosen to accommodate the discrete-time complementarity formulation of the user equilibrium model.

Although Friesz et al. (2011) show that analysis and computation of dynamic user equilibrium with constant travel demand is tremendously simplified by stating it as a differential variational inequality (DVI), they do not discuss how elastic demand may be accommodated within a DVI framework. Friesz and Meimand (2014) later extend the DVI formulation to an elastic demand setting, although that paper does not discuss the existence and the computation of E-DUE, which are our main focus here. Such a DVI formulation for the E-DUE problem is not a straightforward extension. In particular, the DVI presented therein has both infinite-dimensional and finite-dimensional terms. Moreover, for any given origin–destination pair, inverse travel demand corresponding to a dynamic user equilibrium depends on the terminal value of a state variable representing cumulative departures. The DVI formulation achieved in that paper is significant because it allows the still emerging theory of differential variational inequalities to be employed for the analysis and computation of solutions of the elastic-demand DUE problem when simultaneous departure time and route choices are within the purview of users, all of which constitutes a foundation problem within the field of dynamic traffic assignment.

A good review of recent insights into abstract differential variational inequality theory, including computational methods for solving such problems, is provided by Pang and Stewart (2008). Also, differential variational inequalities involving the kind of explicit, agent-specific control variables employed herein are presented in Friesz (2010).

1.2. Discussion of contributions made in this paper

In this paper, we present a unified theory and a general framework for formulating, analyzing, and computing the simultaneous route-and-departure-time (SRDT) dynamic user equilibrium with elastic demand (E-DUE). Such an analytic framework is meant to allow qualitative analyses on E-DUE to be conducted in a rigorous manner, and to accommodate any dynamic network loading model expressible by an embedded effective path delay operator.

We show, using measure-theoretic argument, that a general SRDT E-DUE can be cast as a variational inequality problem. Unlike existing VI formulations in the literature, this VI is defined on an extended Hilbert space, which facilitates the analysis regarding existence and computation. As a result, the existence of E-DUE is formally established in the most general setting; that is, it incorporates both route and departure time choices of travelers, and does so without invoking the *a priori* boundedness of path departure rates.[1]

This paper also makes a significant contribution to the computation of SRDT E-DUE, by proposing three different algorithms and analyzing their convergence conditions. These are achieved through the VI formulation of the E-DUE model. Regarding algorithms and computation, our paper's intent is to: (i) document how far the available mathematics can take us in assuring convergence and (ii) illustrate what can be done computationally when proceeding heuristically by relaxing monotonicity assumptions needed to assure convergence.

In the following sections we will discuss the existence and computation of E-DUE problems in detail, while referring to the work presented in this paper and by other scholars.

1.2.1. Existence of SRDT E-DUE

As commented by Han et al. (2013c), the most obvious approach to establishing existence is to convert the problem to an equivalent variational inequality problem or a fixed-point problem and then apply a version of Brouwer's famous fixed-point existence theorem. Nearly all proofs of DUE existence employ such an existence theorem, either implicitly or explicitly. One statement of Brouwer's theorem appears as Theorem 2 of Browder (1968). Approaches based on Brouwer's theorem

[1] We refer the reader to Han et al. (2013c) for an illustration of the subtlety of the *a priori* bound on the path departure rates when it comes to the SRDT notion of DUEs.