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Optimal layout of transshipment facility locations on an infinite homogeneous plane

Weijun Xie [a], Yanfeng Ouyang [b,*]

[a] School of Industrial & Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, United States
[b] Department of Civil and Environmental Engineering, University of Illinois, Urbana-Champaign, Urbana, IL 61801, United States

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This paper studies optimal spatial layout of transshipment facilities and the corresponding service regions on an infinite homogeneous plane $\mathbb{R}^2$ that minimize the total cost for facility set-up, outbound delivery and inbound replenishment transportation. The problem has strong implications in the context of freight logistics and transit system design. This paper first focuses on a Euclidean plane and shows that a tight upper bound can be achieved by a type of elongated cyclic hexagons, while a cost lower bound based on relaxation and idealization is also obtained. The gap between the analytical upper and lower bounds is within 0.3%. This paper then shows that a similar elongated non-cyclic hexagon shape, with proper orientation, is actually optimal for service regions on a rectilinear metric plane. Numerical experiments are conducted to verify the analytical findings and to draw further insights.

1. Introduction

Problems related to facility location (e.g., fixed-charge location problems) and routing (e.g., travel salesman problem, or TSP) impose two fundamental yet distinct challenges to logistics system design. Due to their intrinsic complexity, these problems are typically handled separately in the literature (e.g., see Daskin, 1995; Toth and Vigo, 2001 for complete reviews). Relatively fewer studies looked at the integrated "*location-routing*" problem with or without inventory considerations (e.g., Perl and Daskin, 1985; Shen and Qi, 2007). The location of distribution centers and the routing of outbound delivery vehicles are optimized simultaneously while inbound shipment (i.e., providing replenishment to these facilities) is assumed to be via direct visits (or more often, omitted from the model). Furthermore, most of these efforts focused on developing discrete mathematical programming models which can only numerically solve very limited-scale problem instances. In particular, little is known about the optimal facility layout, the suitable customer allocation, and the optimal vehicle tour in infinite homogeneous planes. Recently, Cachon (2014) studied a different location-routing model in a homogeneous Euclidean plane which optimizes the spatial layout of facilities that serve distributed customers (i.e., similar to a median problem) and the routing of an inbound vehicle which visits these facilities (i.e., similar to a TSP problem). The model tried to minimize the total cost related to outbound customer access (i.e., direct shipment) and inbound replenishment transportation.

This problem has strong implications on practical logistics systems design in real-world contexts. For example, transshipment is often used in a timber harvesting system, where the lumbers are collected by trucks to local processing mills, and then shipped out by train. Train capacity is generally orders of magnitudes larger than that of local trucks, and it is often

* Corresponding author.
E-mail address: yfouyang@illinois.edu (Y. Ouyang).

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sufficient to assume infinite capacity of a train and design a single train track route for a large area of forest. Another example is commuter transit system design in low-demand areas (e.g., building upon ideas from Nourbakhsh and Ouyang (2012)), where a bus (with sufficient capacity) collects passengers from a certain region at optimally located bus stops (where local passengers gather). Many similar examples can be found in the contexts of planning various mobile service systems; e.g., those related to vending trucks, concert band tours, and others.

Mathematically, this problem can be described as follows (see Fig. 1(a) for an illustration of the notation). We use a transshipment system to serve uniformly distributed customers (with demand density λ per area-time) on an infinite homogeneous Euclidean plane $\mathbb{R}^2$. Transshipment facilities can be constructed anywhere with a prorated set-up and operational cost f per facility-time. All facilities receive replenishment from a central depot, which is co-located at one of the facilities, through an inbound truck with infinite capacity. This truck will supply all facilities along one tour, incurring average transportation cost of C per distance.[1] Without losing generality, we assume that the transshipment facilities are indexed along the tour of the inbound truck; i.e., the inbound truck starts from facility 1 (the depot), visits facilities sequentially in set $\mathcal{N} = \{1, 2, \ldots, N\}$ before returning to facility 1, where $N := |\mathcal{N}| \to \infty$ is the total number of facilities. Facility $i \in \mathcal{N}$ is located at $x_i \in \mathbb{R}^2$ to serve the customers in its service region $\mathcal{A}_i \in \mathbb{R}^2$ through direct shipment, with a transportation cost of c per demand-distance. For each customer at $x \in \mathcal{A}_i$, we use $\|x - x_i\|$ to denote the outbound travel distance. Moreover, for simplicity, the size of service region $\mathcal{A}_i$ is denoted by $A_i = |\mathcal{A}_i|$, and we assume the inbound truck travels a distance of l_i within A_i. Since all cost terms are relative, we further define $\kappa = \frac{c\lambda}{f}$ (area-demand-distance/facility) and $r = \frac{C}{c\lambda}$ to denote the relative magnitudes of facility cost and inbound transportation cost as compared to the outbound cost.

Some basic properties of this problem is readily available. For any given facility layout $\{x_i : i \in \mathcal{N}\}$, each customer should obviously choose the nearest facility for service and a tie may be broken arbitrarily. Thus, the set of service regions $\{\mathcal{A}_i : i \in \mathcal{N}\}$ must form a Voronoi diagram (Okabe et al., 1992; Du et al., 1999), where

$$\mathcal{A}_i = \left\{x \in \mathbb{R}^2 : \|x - x_i\| \leqslant \|x - x_j\|,\ \forall j \neq i\right\}. \tag{1}$$

Moreover, for an optimal TSP tour along N facility locations in the Euclidean plane, it is easy to see that the optimal TSP tour has no crossover between any four facility locations; otherwise, a simple local perturbation can improve the solution. With this, the optimal solution of our problem must satisfy the following properties.

Property 1. *For any given facility locations* $\{x_i : i \in \mathcal{N}\}$*,*

- *the optimal service regions form a Voronoi diagram, i.e.,* (1) *holds and*

$$\bigcup_i \mathcal{A}_i = \mathbb{R}^2; \tag{2}$$

- *each service region $\mathcal{A}_i$ is a convex polygon* (Okabe et al., 1992) *with the number of sides n_i;*
- *for all $x \in \mathcal{A}_i \bigcap \mathcal{A}_j$, $\|x - x_i\| = \|x - x_j\|$, and therefore, for all $i \in \mathcal{N}$,*

$$l_i = \frac{1}{2}(\|x_i - x_{i-1}\| + \|x_i - x_{i+1}\|); \tag{3}$$

- *the maximum distance $\max_{x \in \mathcal{A}_i} \|x - x_i\|$ is achieved by one of the corners of $\mathcal{A}_i$ if the service area A_i is bounded;*
- *the line segment connecting x_i and x_{i+1} is perpendicular to the interception line of $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$.*

From Property 1, we can see that the boundary between any two adjacent facilities extends a triangle with either of the facility locations (e.g., ΔEBD and ΔGBD in Fig. 1(b)). Moreover, these two triangles obviously must be identical (i.e., $\Delta EBD \cong \Delta GBD$). For simplicity, we define the following:

Definition 1. Within each facility service region,

- A "*basic triangle*" is the one extended by the facility location and one side of the service region border; e.g., ΔEBD or ΔGBD in Fig. 1(b).
- A "*basic angle*" is the internal angle of a basic triangle at the facility location; e.g., $\angle BED$ or $\angle BGD$ in Fig. 1(b).

In an infinite plane, a suitable objective is to find the optimal facility layout $\{x_i : i \in \mathcal{N}\}$, the service region partition $\{\mathcal{A}_i : i \in \mathcal{N}\}$, and the inbound truck tour that minimize the total system cost per unit area-time including facility set-up, outbound delivery and inbound replenishment cost per unit area-time. Obviously, the optimal number of facilities $N \to \infty$; otherwise, the average outbound cost goes to infinity since at least one of the facilities would have an unbounded service region on this infinite plane. Hence, the optimization problem can be expressed as follows,

[1] This assumption makes the replenishment frequency irrelevant to the optimization problem.