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Analysis of fixed-time control ☆

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ABSTRACT

The paper presents an analysis of the traffic dynamics in a network of signalized intersections. The intersections are regulated by fixed-time (FT) controls, all with the same cycle length or period, T. The network is modeled as a queuing network. Vehicles arrive from outside the network at entry links in a deterministic periodic stream, also with period T. They take a fixed time to travel along each link, and at the end of the link they join a queue. There is a separate queue at each link for each movement or phase. Vehicles make turns at intersections in fixed proportions, and eventually leave the network, that is, a fraction $r(i,j)$ of vehicles that leave queue i go to queue j and the fraction $[1 - \sum_j r(i,j)]$ leave the network. The storage capacity of the queues is infinite, so there is no spill back. The main contribution of the paper is to show that if the signal controls accommodate the demands then, starting in any initial condition, the network state converges to a unique periodic orbit. Thus, the effect of initial conditions disappears. More precisely, the state of the network at time t is the vector $x(t)$ of all queue lengths, together with the position of vehicles traveling along the links. Suppose that the network is stable, that is, $x(t)$ is bounded. Then

(1) there exists a unique periodic trajectory x^*, with period T;
(2) every trajectory converges to this periodic trajectory;
(3) if vehicles do not follow loops, the convergence occurs in finite time.

The periodic trajectory determines the performance of the entire network.

1. Introduction

Traffic in an urban network is determined by intersection signal control and the pattern of demand. The movement of vehicles is often modeled as a queuing network as, for example, in Papageorgiou et al. (2003) and Mirchandani and Head (2001). Roughly speaking, a vehicle arrives from outside the network at an entry link; travels along a link at a fixed speed; at the end of the link it arrives at an intersection and joins a queue of vehicles for the next link in its path; the queue is served at a specified saturation flow rate when that movement is actuated by the signal; eventually the vehicle leaves the network.

In the U.S. 90% of traffic signals follow fixed time (FT) controls, which operate the signal in a fixed periodic cycle, independent of the traffic state (Federal Highway Administration., 2008). Despite its practical importance, little attention has

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been paid to understanding how traffic behaves under under FT control. Published work has studied queues at a single, isolated intersection, as in Miller (1963). The steady state optimal control of single intersections is studied in Improta and Cantarella (1984), Haddad et al. (2014), Gazis (2002). The latter work derives the optimal control settings required to minimize different objectives including queuing delays, but does not address the effect of initial conditions on solution trajectories or their convergence. Gazis (1964, 2002) analyze oversaturated intersections and Varaiya (2013) inroduces an adaptive control for undersaturated networks. But neither work analyzes the behavior of solution trajectories. Signal timing tools used by traffic engineers often employ empirical models (Webster, 1958; Transportation Research Board, 2010) in combination with simulations, assuming steady state conditions. But the absence of theory establishing convergence to a unique steady state calls into question whether the traffic flows achieve the performance for which these signals are tuned.

We analyze vehicle movement under two assumptions: first, all the signals have a fixed time (FT) control with the same cycle time or period T; second, vehicles from outside enter the network in periodic streams with the same period. Periodic demands include constant demands, which is the assumption in commercial packages used to design FT controls. Also, if there are intersections with FT controls with different cycles $T_1, \ldots, T_k$, they are all periodic with the same period $T = \text{lcm}\{T_1, \ldots, T_k\}$.

The state of the signalized network at any time t consists of $x(t)$, the vector of all queue lengths, together with the position of all vehicles that are traveling along a link but have not yet reached a queue. A queue increases when vehicles arrive and decreases when the control serves that queue. We treat time as continuous and vehicles as a fluid instead of as discrete entities. As a result the evolution of the network is described by a delay-differential equation, in which the delay comes from the travel time of a vehicle as it moves from one queue to the next. In an actual transportation network, the arrival and service processes are stochastic. However, an exact analysis of queue-length processes in a stochastic queueing network is very difficult, if not impossible; except for very simple examples such as an isolated intersection, the underlying Markov chain of the system is intractable. Therefore, we consider deterministic arrival and service processes in this paper.

From a traffic theory viewpoint, our main contribution is to show that there is a unique periodic trajectory $x^*(t)$ of the queue length vector to which every trajectory $x(t)$ converges; moreover, in case individual vehicles do not circulate in loops, the convergence is in finite time. The periodic orbit of course determines every possible performance measure, such as delay, travel time, amount of wasted green, and signal progression quality, see Day et al. (2014). An outstanding open problem is to calculate this periodic orbit without simulation. If this can be done, one would have a computational procedure to design the FT control for a network that optimizes any performance measure.

The results have some independent mathematical interest. The delay-differential equation is not Lipschitz, and existence and uniqueness of a solution is established using the reflection map of queuing theory (Harrison and Reiman, 1981; Whitt, 2001). The differential equation is periodic (with period T), and the existence of a periodic orbit is proved using the Poincare map. The global stability of this periodic orbit depends on a monotonicity property reminiscent of that in freeway models (Gomes et al., 2008).

The rest of the paper is organized as follows. Section 2 presents the main results for a single queue. Section 3 describes the basic results for the network model. Results for the case of periodic demand and FT control are presented in Section 4. The main conclusion and some open questions are summarized in Section 5.

2. Single queue without routing

Time is continuous, $t \geqslant 0$. The length or size of a single queue $x(t),\ t \geqslant 0$, evolves as

$$\dot{x}(t) = e(t) - b(t), \tag{1}$$

with arrivals $e(t) \geqslant 0$, departures $b(t),\ t \geqslant 0$, and initial queue $x(0) = x_0 \geqslant 0$. Arrivals $e(t)$ are exogenously specified. There is a specified saturation flow or service rate $c(t) \geqslant 0,\ t \geqslant 0$, so departures are given by

$$b(t) = \begin{cases} c(t), & \text{if } x(t) > 0, \\ \in [0, c(t)], & \text{if } x(t) = 0, \\ 0, & \text{if } x(t) < 0. \end{cases} \tag{2}$$

Express the departure process as

$$b(t) = c(t) - y(t), \quad t \geqslant 0, \tag{3}$$

so $y(t)$ is the rate at which service is unused. From (2),

$$y(t) \geqslant 0, \text{ and } x(t)y(t) \equiv 0.$$

Rewrite (1) as

$$\dot{x}(t) = [e(t) - c(t)] + y(t),$$

or in functional form as

$$x = u + v, \tag{4}$$