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Traffic user equilibrium and proportionality

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ABSTRACT

We discuss the problem of proportionality and uniqueness for route flows in the classical traffic user equilibrium model. It is well-known that under appropriate assumptions the user equilibrium (f, x) is unique in the link flow x but typically not in the route flow f. We consider the concept of proportionality in detail and re-discuss the well-known relation between the so-called bypass proportionality and entropy maximization. We exhibit special proportionality conditions which uniquely determine the route flow f. The results are illustrated with some simple example networks.

1. Introduction

In classical user equilibrium models under appropriate assumptions only the link flow x of a user equilibrium (UE) (f, x) is uniquely determined by the equilibrium conditions but in general not the route flow f. As uniqueness is a desirable property for various reasons (*cf.* Section 2), researchers are interested in models that lead to an equilibrium (f, x) which is unique and stable also in f.

A natural way to achieve that is to compute the (unique) flow f which maximizes the so-called entropy. An alternative way is to directly look for extra conditions defining a unique solution f. In this paper we follow the latter approach. The extra conditions to be imposed should not only fix a unique route flow f but they should also have a practical interpretation. Here an important role is played by proportionality conditions.

The aim of this paper is to rediscuss the concept of proportionality and to obtain an "exact" set of (proportionality) conditions which determine the user equilibrium route flow f in a unique and stable way.

We give a short overview of related work. Rossi et al. (1989) were the first to propose entropy maximization in order to obtain a unique route flow. Bar-Gera and Boyce (1999) then studied the implication of route flow entropy maximization and among others show that the corresponding flow satisfies the so-called by-pass proportionality conditions. In Bar-Gera (2006) Bar-Gera studies the structure of all possible UE route flows and investigates the properties of entropy maximizing flows in particular proportionality. A primal method for computing this flow is proposed and tested on a set of real world networks.

In 2010, Bar-Gera (2010) came up with a new algorithm for computing user equilibria. His traffic assignment by paired alternative segments (TAPAS) depends on the computation of the origin based user equilibrium link flows. The algorithm iterates alternatingly towards user equilibrium and entropy maximization. The paper reports on numerical experiments.

The subsequent article (Bar-Gera et al., 2012) compares TAPAS and two other traffic assigment tools with respect to proportionality. In Florian and Morosan (2014), Florian and Morosan have shown through numerical experiments that the Frank

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Wolfe algorithm as well as TAPAS generate UE route flows which nearly satisfy proportionality. It also presents conditions assuring that a step of the Frank Wolfe algorithm increases the entropy.

There are many articles on stability and sensitivity of UE flows, e.g., Tobin and Friesz (1988). In Lu and Nie (2010), the authors investigate the uniqueness and stability of UE link flows as well as the uniqueness of corresponding route flows f maximizing appropriate (strictly convex) functions of f such as, e.g., the entropy function. Also Bar-Gera et al. (2013) deals with sensitivity of UE link flows depending on different network design parameters.

Kumar and Peeta (2015) present and investigate an entropy weigthed averaging method as a generalization of entropy maximization. They prove that under standard assumtions also this extended method yields a unique route flow. For further literature we refer the reader to the papers cited in the articles mentioned above.

The present paper is most closely related to the study in Bar-Gera (2006) who introduces the notion of n-consistent route sets and analyses the structure of the route flows f that are feasible for the entropy maximization program. The approach in Bar-Gera (2006) is based on a primal method for computing the entropy maximizing route flow f. By contrast, in the present paper we seek to directly obtain an appropriate set of extra conditions which uniquely fix f. Our paper can be seen as a complement to Bar-Gera (2006) in the sense that we approach the uniqueness question for f from another perspective, thereby obtaining some of the results in Bar-Gera (2006) in a more direct way. We therefore will regularly compare and relate our arguments to the results in Bar-Gera (2006).

Our paper is organized as follows. After a preliminary Section 2, the concept of proportionality is treated in detail in Section 3.1. Motivated by Bar-Gera (2006), we introduce the notion of proportionality conditions of order n. In Section 3.2 we review the relations between entropy maximization and so-called by-pass proportionality conditions from Bar-Gera and Boyce (1999). In Section 3.3 we determine proportionality conditions that uniquely determine the route flows f. The results are illustrated with some simple example networks which we also analyze numerically. Section 4 studies the algebraic structure of the proportionality conditions presented in Section 3.3 and relates them (at least for networks with only one OD-pair) to certain pairs of routes.

2. Preliminaries

We start with a short introduction into the classical Wardrop-Beckmann traffic equilibrium model. Given a directed traffic network $N = (V, E)$ with node set V, directed link set E and *origin–destination pairs* (O–D pairs for short) $(o_w, t_w) \in V \times V$ with corresponding *demands* $d_w \geqslant 0, w \in W$, we let $\mathcal{R}_w$ denote the set of simple directed o_w–t_w routes, and $\mathcal{R} := \bigcup_{w\in W}\mathcal{R}_w$. A *traffic flow* for the given demand $d \in \mathbb{R}_+^{|W|}$ is a pair of vectors (f, x) of the form

$$\Lambda f = d, \quad x = \Delta f, \quad f \geqslant 0, \tag{2.1}$$

where the component $x_e, e \in E$ of $x \in \mathbb{R}^{|E|}$ is the flow on link $e \in E$ and the components f_p of $f \in \mathbb{R}^{|\mathcal{R}|}$ indicate the amount of flow on a route $r \in \mathcal{R}$. The elements of the matrices $\Delta \in \mathbb{R}^{|E|\times|\mathcal{R}|}, \Lambda \in \mathbb{R}^{|W|\times|\mathcal{R}|}$ are defined by

$$\Delta_{er} = \begin{cases} 1 & e \text{ is an edge of } r \\ 0 & \text{otherwise} \end{cases} \qquad \Lambda_{wr} = \begin{cases} 1 & r \in \mathcal{R}_w \\ 0 & \text{otherwise} \end{cases}.$$

We call x the *link flow* and f the *route flow* of (f, x).

Let $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ be non-negative, continuous, and non-decreasing link cost functions, defining (separable) *costs* or *travel times* $c_e(x_e), e \in E$, for a given traffic flow $x \in \mathbb{R}_+^{|E|}$. We say that $x \in \mathbb{R}^{|E|}$ *induces* the link costs $c_e(x_e), e \in E$ and route costs $c_r(x) = \sum_{e\in r} c_e(x_e),\ r \in \mathcal{R}$.

Definition (*Wardrop Equilibrium*). Consider a traffic flow $(\bar{f}, \bar{x})$ as in (2.1) with corresponding induced costs $c_e(\bar{x})$. Then $(\bar{f}, \bar{x})$ is called a *Wardrop equilibrium* (or *user equilibrium* (UE)) if for all $r, q \in \mathcal{R}_w, w \in W$ the following condition is satisfied:

$$\bar{f}_r > 0 \quad \Rightarrow \quad \begin{cases} c_r(\bar{x}) = c_q(\bar{x}) & \text{if } f_q > 0 \\ c_r(\bar{x}) \leqslant c_q(\bar{x}) & \text{if } f_q = 0 \end{cases}$$

Thus, in particular, in a Wardrop Equilibrium all *used* o_w–t_w routes have equal costs. A well-known equivalent characterization is due to Beckmann:

Lemma 2.1. *The following are equivalent for a traffic flow* $(\bar{f}, \bar{x})$*:*

(1) *The flow* $(\bar{f}, \bar{x})$ *is a Wardrop equilibrium.*
(2) *[Beckmann's formulation]* $(\bar{f}, \bar{x})$ *is solution of the (convex) program:*

$$\min_{x,f} z(x) := \sum_{e\in E} \int_0^{x_e} c_e(\tau) d\tau \qquad \text{s.t.} \qquad \begin{aligned} &\Lambda f = d \\ &\Delta f - x = 0 \\ &f \geqslant 0 \end{aligned} \tag{2.2}$$