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# A semi-analytical approach for solving the bottleneck model with general user heterogeneity

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ABSTRACT

This paper proposes a novel semi-analytical approach for solving the dynamic user equilibrium (DUE) of a bottleneck model with general heterogeneous users. The proposed approach makes use of the analytical solutions from the bottleneck analysis to create an equivalent assignment problem that admits closed-form commute cost functions. The equivalent problem is a static and asymmetric traffic assignment problem, which can be formulated as a variational inequality problem (VIP). This approach provides a new tool to analyze the properties of the bottleneck model with general heterogeneity, and to design efficient solution methods. In particular, the existence and uniqueness of the DUE solution can be established using the P-property of the Jacobian matrix. Our numerical experiments show that a simple decomposition algorithm is able to quickly solve the equivalent VIP to high precision. The proposed VIP formation is also extended to address simultaneous departure time and route choice in a single O–D origin-destination network with multiple parallel routes.


## 1. Introduction

The analysis of dynamic commute travel patterns, originated by Vickrey (1969) and refined subsequently in Hendrickson and Kocur (1981), Smith (1984), Daganzo (1985), Newell (1987), Arnott et al. (1990, 1994), Yang and Huang (1997) has been extensively studied in the literature. The underlying assumption in these analyses is that travelers make trade-off between the anticipated costs of travel time and schedule delay (incurred when travelers cannot arrive at their destination at a desired time). Accordingly, the general pattern of commuters' departure time choices is explained as a dynamic user equilibrium at which nobody can reduce his/her own commute cost by unilaterally shifting his/her route and departure time choice. Vickrey's original bottleneck model does not consider users who differ in their valuation of travel time and schedule delay. Such heterogeneity has since attracted much attention because of its potential impacts on the equilibrium solutions as well as the welfare effects of demand management policies (Small, 1982; Cohen, 1987; Arnott et al., 1994; Lindsey, 2004; Small et al., 2005; van den Berg and Verhoef, 2011b; Liu and Nie, 2011; Hall, 2013).

It is well known that analytical solutions for the bottleneck model with user heterogeneity exist in special cases, such as when restrictions are imposed on how the value of time ($\alpha$) and unit schedule cost ($\beta$ for early arrival and $\gamma$ for late arrival) may be correlated. Vickrey (1973) studied the case where $\alpha$ is proportional to $\beta$ and $\gamma$. Cohen (1987) considered two typical groups of commuters: low-income commuters who have low value of time but rigid work schedule and high-income

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commuters who value their time higher and have more flexible work schedule. His analysis requires the ratio of $\beta/\gamma$ to be a constant. Arnott et al. (1988, 1994) generalized Cohen's analysis by also considering other dimensions of user heterogeneity (e.g., desired arrival time). van den Berg and Verhoef (2011a) assume that $\beta$ and $\gamma$ are fixed and identical for all users while $\alpha$ varies across users. Palma and Lindsey (2002a) assume that $\alpha$ is log-normally distributed and all the users have the same ratios of $\beta/\alpha$ and $\gamma/\beta$. van den Berg and Verhoef (2011b) examined welfare effects of Vickrey's time-varying toll using a similar heterogeneity structure as Cohen's. They generalize the model to handle arbitrary number of user groups. Qian and Zhang (2013) studied the morning commute problem with infinite number of user groups by also assuming the constant ratio of $\beta$ over $\gamma$ and extended the model into two-parallel routes network. Recently, Hall (2013) allows user preferences to be continuously distributed in all three dimensions, but the restriction of constant $\beta/\gamma$ ratio is still imposed. The heterogeneity setting of Cohen (1987) was also adopted to analyze simple network bottleneck model with two parallel routes by Arnott et al. (1992) and Liu and Nie (2011).

Several studies have addressed the analytical properties of the bottleneck model, such as the existence and the uniqueness of the equilibrium solution (e.g., Hendrickson and Kocur, 1981; Smith, 1984; Newell, 1987; Daganzo, 1985; Lindsey, 2004). Most of these studies considered user heterogeneity only in the dimension of the desired arrival times, with the exception of Newell (1987) and Lindsey (2004). Newell also made use of the assumption that the ratio of $\beta$ and $\gamma$ is constant to simplify his analysis. Lindsey (2004) considered a bottleneck model with a general heterogeneity structure (i.e., all user preferences are allowed to vary independently) and proved that it admits one and only one user equilibrium under mild conditions. His result is theoretically significant, albeit it does not prescribe a solution method for the dynamic user equilibrium under general heterogeneity. In fact, if a general joint distribution of user preferences is considered, obtaining an equilibrium solution for bottleneck model seems analytically intractable.

Vickrey's model also inspired a large body of literature under the umbrella of dynamic traffic assignment (DTA) (e.g., Merchant and Nemhauser, 1978; Friesz et al., 1993; Ran et al., 1993; Lu et al., 2006; Nie and Zhang, 2007), which seeks to forecast equilibrium traffic patterns in more general network settings. Because the DTA models aim at representing realistic traffic phenomena (e.g., physical queue, traffic controls), the commute cost is typically evaluated through traffic simulation (also known as dynamic network loading) instead of closed-form formulae. Accordingly, the equilibrium problem may only be solved approximately in most cases. The reader is referred to Peeta and Ziliaskopoulos (2001) for a comprehensive review of the DTA literature. Recently, Ramadurai et al. (2010) and Pang et al. (2012) tackled the Vickrey's bottleneck model with heterogeneous users using a DTA approach. Specifically, they formulated and solved the problem as a general linear complementarity system, in which another important dimension of travel preferences, i.e., the desired arrival time, is considered. Note that time is discretized in Ramadurai et al. (2010). Later, Pang et al. (2012) proposed to use the time-stepping numerical technique to approximate the discrete-time model.

The approach proposed in this paper differs from those in the classical bottleneck analysis and the DTA research. It makes use of the analytical solutions from the bottleneck analysis to create an equivalent assignment problem that explicitly admits closed-form commute cost functions. More specifically, the underlying equivalent assignment problem is a static and asymmetric traffic assignment problem, which may be formulated as a variational inequality problem. We call this approach "semi-analytical" because it blends analytical and numerical methods. This allows us to analyze the analytical properties of the underlying problem since we have a closed-form commute cost function. In particular, we prove the existence and the uniqueness of DUE by examining the P-property of VIP's Jacobian matrix. Perhaps more important for practical purposes, the proposed VIP can be solved to high precision with simple assignment algorithms, which makes it a useful instrument to perform numerical analysis for congestion management policies using bottleneck model with general user heterogeneity.

For the reminder, Section 2 introduces a single bottleneck model with a fixed number of heterogeneous commuters. In Section 3, a semi-analytical approach is developed to transform the DUE problem of the bottleneck model into a static traffic assignment problem, which is then formulated as a variational inequality problem. Section 4 proves the existence and the uniqueness of DUE solution. In Section 5, we extend the variational inequality formulation to solve DUE in a single origin–destination corridor network with multiple parallel routes. Section 6 reports results of several numerical experiments, including one constructed using empirical data from California State Route 91. Section 7 concludes our findings.

## 2. Model setting

Consider a fixed number of travelers who commute from home to work through a corridor during the morning peak-hour. Without loss of generality, we assume that a bottleneck is located at the exit of the corridor, such as an off-ramp leading to downtown. When the demand (the departure rate) exceeds the capacity of the bottleneck, denoted as $s$, a queue forms and consequently commuters experience queuing delays. Therefore, the travel time along the corridor consists of two parts: (1) the fixed free flow travel time $T$, i.e., the time needed to traverse the corridor when there is no congestion, and (2) queuing delay. Since $T$ does not affect the analysis in the case of single bottleneck, it is assumed to be zero except in Section 5 where the route choice is discussed. Note that travel time and queuing delay are equivalent when $T = 0$. When commuters arrive at their work place, their schedule delay will be the difference between the actual and desired arrival times. Each commuter chooses a departure time $t$ (from the bottleneck) so as to minimize his/her *commute cost* $c(t)$, which consists of the costs of travel delay and schedule delay as in the classic bottleneck analysis (Vickrey, 1969). Specifically, the following piecewise linear function is adopted in this paper