Transportation Research Part B 70 (2014) 163–172

Contents lists available at ScienceDirect

Transportation Research Part B

journal homepage: www.elsevier.com/locate/trb

Benders Decomposition for Discrete–Continuous Linear Bilevel Problems with application to traffic network design

Pirmin Fontaine *, Stefan Minner

Logistics & SCM, TUM School of Management, Arcisstraße 21, 80333 Munich, Germany

ARTICLE INFO

Article history:
Received 19 December 2013
Received in revised form 9 September 2014
Accepted 10 September 2014

Keywords:
Bilevel Programming
Discrete Network Design Problem
Benders Decomposition
Linear approximation

ABSTRACT

We propose a new fast solution method for linear Bilevel Problems with binary leader and continuous follower variables under the partial cooperation assumption. We reformulate the Bilevel Problem into a single-level problem by using the Karush–Kuhn–Tucker conditions. This non-linear model can be linearized because of the special structure achieved by the binary leader decision variables and subsequently solved by a Benders Decomposition Algorithm to global optimality. We illustrate the capability of the approach on the Discrete Network Design Problem which adds arcs to an existing road network at the leader stage and anticipates the traffic equilibrium for the follower stage. Because of the non-linear objective functions of this problem, we use a linearization method for increasing, convex and non-linear functions based on continuous variables. Numerical tests show that this algorithm can solve even large instances of Bilevel Problems.

1. Introduction

As we are facing increasing population in cities, the demand for transportation increases. This leads to more congested roads and longer travel times. Moreover, congestions lead to air pollution, noise pollution and a lower quality of living. Therefore, traffic networks have to be expanded and an efficient usage of the budget in network expansions should be achieved. In the literature, these problems are addressed as Bilevel Problems (i.e., LeBlanc, 1975; Farahani et al., 2013; Gao et al., 2005; Poorzahedy and Turnquist, 1982; Luathep et al., 2011).

Bilevel Problems are mathematical programming problems consisting of an optimization problem (the upper one – also called the leader) with nested optimization problems (the lower ones – also called follower) in the constraints. In practice, these problems can occur if decentralized or hierarchical decisions have to be taken (Schneeweiß, 2003). First, the leader has to decide over a subset of the decision variables, which affects the feasible region of the follower. Afterwards, the follower has to decide over the other subset of decision variables, which affects the objective value of the leader. Ben-Ayed and Blair (1990) showed that even Linear Bilevel Problems with continuous leader and follower decision variables are $\mathcal{NP}$-hard. In this paper, we consider the linear case with binary leader variables and continuous follower variables (DCBLP). Over the past decades, several methods have been proposed to solve Linear Bilevel Problems (BP). A survey on different solution methods and applications on Bilevel Problems can be found in Colson et al. (2005). Bard and Moore (1990) presented a branch-and-bound approach based on the Kuhn-Tucker conditions of the follower problem and Hansen et al. (1992) proposed a branch-and-bound algorithm which branches on binding follower constraints. Both algorithms were introduced for the continuous BP, but can also be applied for BP with integer leader variables. Recently, Saharidis and Ierapetritou (2009) suggested a

* Corresponding author.
E-mail address: pirmin.fontaine@tum.de (P. Fontaine).

http://dx.doi.org/10.1016/j.trb.2014.09.007

Benders Decomposition (BD) approach for solving the mixed-integer BP with discrete leader variables and continuous follower variables. However, this approach still has to solve a mixed-integer program (MIP) in the Master Problem and in the Slave Problem, because it solves a Bilevel Problem in the Slave Problem by applying the Active Constraint Strategy (Grossmann and Floudas, 1987). In this paper, we introduce a new solution method for the BP with binary leader variables, which is also using BD, but only solves continuous Slave Problems.

One application of Bilevel Programming are Urban Network Design Problems. These problems improve an existing traffic network by adding new links to the network or increasing the capacity of already existing links. The latter problem is called the Continuous Network Design Problem (CNDP) (Abdulaal and LeBlanc, 1979) and the former, which we address in this paper, the Discrete Network Design Problem (DNDP) (LeBlanc, 1975). All these problems have to be modeled as Bilevel Problems, as the objective of the network designer – to reduce congestion in the network – and the objective of the follower – to find the fastest way from origin to destination – do not have to be the same (Braess et al., 2005). Farahani et al. (2013) recently summarized the different existing models and solution methods.

Abdulaal and LeBlanc (1979) proposed a direct search method for the CNDP and Suh and Kim (1992) presented a descent algorithm for non-linear Bilevel Problems. Moreover, Simulated Annealing and Genetic Algorithms were used to solve the CNDP (Meng and Yang, 2002; Xu et al., 2009; Mathew and Sharma, 2009).

For the DNDP, which is more difficult to solve than the CNDP, (LeBlanc, 1975) proposed a branch-and-bound method. A Generalized Benders Decomposition approach with the use of support functions was proposed by Gao et al. (2005).

Luathep et al. (2011) presented a mixed-integer linear formulation for the DNDP which formulates the follower problem with the variational inequality problem. This formulation approximates the travel time function with piecewise linear terms and relaxes bilinear terms by introducing binary auxiliary variables. Farvaresh and Sepehri (2011) applied the Karush–Kuhn–Tucker conditions to transform the problem into a non-linear MIP. This formulation was further linearized by introducing binary auxiliary variables. Recently, Wang et al. (2013) presented a global optimization approach which uses the relation between the system optimum and the user optimum. Further, Ekström et al. (2012) proposed a global optimization mixed-integer program for a similar problem, the Toll Location Problem.

Since real-size instances are still difficult to solve, approximation algorithms and several heuristics were applied. Poorzahedy and Turnquist (1982) proposed branch-and-bound based heuristic, Poorzahedy and Abulghasemi (2005) applied the Ant Colony System metaheuristic and Poorzahedy and Rouhani (2007) showed that hybrid metaheuristics based on tabu search, simulated annealing and genetic algorithm perform even better on the DNDP.

Besides the contribution to solve considerably larger instances of DCBLPs, we present a formulation of the Discrete Network Design Problem which approximates the non-linear convex objective functions only by piecewise linear terms (without additional binary variables) and can be solved by our algorithm. Compared to Luathep et al. (2011) and Farvaresh and Sepehri (2011), we avoid introducing binary auxiliary variables and the relaxation of bilinear terms. Because of the very small number of binary variables, the linear MIP formulation of the DNDP has computational benefits and we can solve even large instances for the DNDP. We further show how to accelerate the run time of the Slave Problem.

The remainder of this paper is structured as follows. In Section 2, we introduce the general Linear Bilevel Problem and our algorithm. Section 3 introduces the bilevel formulation of the DNDP and shows the linearization. In Section 4, we evaluate the performance of the algorithm on several instances and end with a summary of the proposed procedure, results and outline some future research opportunities.

2. Bilevel Problem and Algorithm

Section 2.1 shows the transformation of the DCBLP into a single-level linear MIP and Benders Decomposition is applied in Section 2.2.

2.1. Transformation to a single-level problem

In the following, we introduce the general formulation for the DCBLP. The leader variables are given by y_i for all $i \in I$ with I the corresponding set of indices and the follower variables by x_j for all $j \in J$ with J the corresponding set. (1) shows the leader objective function with $f'_i \in \mathbb{R}$ for all $i \in I$ and f_j for all $j \in J \in \mathbb{R}$ the objective coefficients. The follower problem is represented by (2)–(4) with the follower objective function in (2) and the follower objective coefficients $c_j \in \mathbb{R}$ for all $j \in J$. K is the set of follower constraints in (3), where each constraint $k \in K$ is defined by its coefficients $a_{kj} \in \mathbb{R}$ for all $i \in I, a'_{kj} \in \mathbb{R}$ for all $j \in J$ and the right hand side b_k. For simplification, we omit constraints in the leader main problem, but the following transformations can all be applied to the more general formulation. Moreover, we assume the partial cooperation assumption (Dempe, 2002; Bialas and Karwan, 1984) – also called an optimistic DCBLP. This allows the leader to select an optimal follower decision among all optimal follower decisions if there exists more than one.

$$\min_y z_L(y,x) = \sum_{i \in I} f'_i y_i + \sum_{j \in J} f_j x_j \tag{1}$$

$$\text{s.t. } \min_x z_F(x) = \sum_{j \in J} c_j x_j \tag{2}$$