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Cost scaling based successive approximation algorithm for the traffic assignment problem

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ABSTRACT

This paper presents a cost scaling based successive approximation algorithm, called ε-BA (ε-optimal bush algorithm), to solve the user equilibrium traffic assignment problem by successively refining ε-optimal flows. As ε reduces to zero, the user equilibrium solution is reached. The proposed method is a variant of bush-based algorithms, and also a variant of the min-mean cycle algorithm to solve the min-cost flow by successive approximation. In ε-BA, the restricted master problem, implying traffic equilibration restricted on a bush, is solved to ε-optimality by cost scaling before bush reconstruction. We show that ε-BA can reduce the number of flow operations substantially in contrast to Dial's Algorithm B, as the former operates flows on a set of deliberately selected cycles whose mean values are sufficiently small. Further, the bushes can be constructed effectively even if the restricted master problem is not solved to a high level of convergence, by leveraging the ε-optimality condition. As a result, the algorithm can solve a highly precise solution with faster convergence on large-scale networks compared to our implementation of Dial's Algorithm B.

1. Introduction

The traffic assignment problem (TAP) lies at the core of transportation planning. The user-equilibrium (UE) TAP has had widespread applications in the last five decades since Beckmann formulated it as a mathematical programming problem (Beckmann et al., 1956). The basic model continues to be widely used, and finding fast algorithms to solve the TAP is of sustained interest to both researchers and practitioners world-wide.

Algorithms to solve the TAP typically operate in the space of link flows (Frank and Wolfe, 1956; LeBlanc et al., 1975), and route flows (Dafermos, 1968; Florian et al., 2009; Jayakrishnan et al., 1994; Larsson and Partriksson, 1992). It has long been known that the TAP is equivalent to a min-cost multi-commodity flow problem on an uncapacitated network with a convex, continuously differentiable objective function, where each commodity is characterized as originating from a single origin. Thereby, algorithms (Nguyen, 1974; Petersen, 1975) have been proposed to decompose the master problem, by solving the single commodity subproblems repeatedly until all of the commodities are optimized simultaneously. Such algorithms have been characterized in recent years as "origin-based".

Origin-based algorithms have recently garnered significant focus in the TAP context due to a family of bush-based algorithms (Bar-Gera, 2002; Dial, 2006; Nie, 2010, 2012). These algorithms maintain an acyclic subnetwork, labeled a bush (Dial, 2006), at each origin and restrict origin-based assignments only to those bushes. They repeatedly solve two subproblems:

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bush equilibration and bush reconstruction (Nie, 2010). The bush equilibration subroutine solves for the UE flow restricted on a bush, also called the restricted master problem (RMP). The acyclicity of a bush is a key building block in the design of a fast algorithm in this family, which allows the fastest possible shortest path tree and, perhaps more importantly, the longest path tree computations. Hence, the bush equilibration can be performed trivially by shifting flows between the longest and shortest paths (Dafermos, 1971; Dial, 2006). In our opinion, a formidable challenge for bush-based algorithms that may preclude very precise convergence (for example, at a relative gap of 10^{-12}) on large-scale networks is how to reconstruct bushes. Because of the requirement of acyclicity for a bush, a non-bush arc that violates optimality conditions may not be appropriately manipulated in bush reconstruction due to the likelihood of acyclicity violation (see Section 3.1). As a result, a bush algorithm may not approach convergence if the bush reconstruction subproblem does not converge to the optimal bush, unless a method is explicitly developed to resolve this issue.

In this paper we present an algorithm called ε-optimal[1] bush algorithm (ε-BA) to solve the TAP by successive approximation. The method is inspired by the min-mean cycle method (Goldberg and Tarjan, 1989) and the successive approximation algorithms for solving the min-cost flow (MCF) problem (Bland and Jensen, 1992; Goldberg and Tarjan, 1990; Rock, 1980). It modifies the RMP in a bush algorithm by operating flows only on a set of selected cycles[2] whose negative mean-costs are small, rather than all negative cycles. A min-mean cycle can be found in $O(|N||A|)$ time (Karp, 1978), which can be fairly expensive. Hence, a cost scaling method is applied to identify a set of cycles with negative mean-costs at most as a predefined threshold value $-\lambda$. The essence of ε-BA is to combine two components: (1) to operate flows on a set of selected cycles whose mean-costs are negatively small; and (2) to solve the set of selected cycles by leveraging the acyclicity of a bush. The proposed algorithm is a variant of bush-based algorithms, and also a variant of the min-mean cycle algorithm. We show that ε-BA could lead to a significantly reduced number of flow operations. In ε-BA, we solve the RMP to ε-optimality before proceeding to the bush reconstruction, and successively refine ε-optimality by reducing ε. As $\varepsilon \downarrow 0$ (ε reduces to zero), ε-BA reduces to a bush algorithm as in Dial (2006) or Nie (2010). We show that ε-BA has the following merits compared to a bush algorithm: (i) ε-BA provides a benchmark indicating when to stop solving the RMP while the RMP is not solved precisely; (ii) given ε-optimality of a bush, ε-BA provides a new rule to reconstruct bushes warranting acyclicity while not necessarily solving the RMP very precisely; (iii) as the key finding of the study, the computational effort of ε-BA can be significantly less than that for a bush algorithm; and (iv) ε-BA can produce a highly precise solution with faster convergence than a bush algorithm, especially on large-scale networks.

The remainder of the paper is organized as follows. Section 2 provides some preliminaries in terms of notation and definitions for the problem. Section 3 provides an overview of the ε-BA. Section 4 presents the cost scaling method to refine ε-optimality restricted on a bush. Section 5 discusses several implementation issues. Section 6 presents numerical experiments to demonstrate the computational performance and convergence properties of the proposed algorithm. Finally, some concluding comments are provided in Section 7.

2. Preliminaries

2.1. Notation

A transportation network $G = (N,A)$ consists of a set of nodes N and a set of directed arcs A (also called links). Suppose G is strongly connected, that is, at least one route exists between any two nodes in G. Flow assigned on arc $(i,j) \in A$ is denoted by x_{ij}. Each arc $(i,j) \in A$ is associated with an unbounded capacity denoted by u_{ij}, and a nonnegative flow-dependent cost (or travel time) $c_{ij}(x_{ij})$, or simply c_{ij} for notational convenience. N consists of a set of origins denoted by $\mathcal{P}$, a set of destinations denoted by $\mathcal{Q}$, and a set of intermediate nodes. A set of origin–destination (O–D) pairs is denoted by a vector $V := \{(p,q)|p \in \mathcal{P}, q \in \mathcal{Q}\}$. In the origin-based TAP, the set of destinations associated with an origin $p \in \mathcal{P}$ is denoted by $\mathcal{Q}(p)$. Demand from each origin $p \in \mathcal{P}$ to its destination $q \in \mathcal{Q}(p)$ is denoted by b_{pq}. Denote the total demand of origin p by B_p, where $B_p = \sum_{q \in \mathcal{Q}(p)} b_{pq}$. Denote by $I(i)$ the set of inbound arcs ending at node i, and $O(i)$ the set of outbound arcs incident from node i.

A walk visits a list of nodes $i_1,\ldots,i_n$ such that for each $k = 1,\ldots,n-1$ there is $(i_k,i_{k+1}) \in A$ or $(i_{k+1},i_k) \in A$. The walk is directed if $(i_k,i_{k+1}) \in A$ for each $k = 1,\ldots,n-1$. A cycle, denoted by W, is a walk where $i_1 = i_n$ and $n > 2$. A directed cycle is a directed walk with the same condition of a cycle. $|W|$ denotes the length (cardinality) of W, or the number of arcs contained in W. A subgraph is a graph (N',A') such that $N' \subseteq N$ and $A' \subseteq A$. In this paper, an arc (i,j) with $x_{ij} > 0$ is called an active arc, and with $x_{ij} = 0$ is called an inactive arc.

A cycle W has an orientation (also called direction), which is either clockwise or counter-clockwise. The set of arcs in W that have the same direction as the cycle is called the set of forward arcs, denoted by $F(W)$. The set of arcs that has the reverse direction as that of the cycle is called the set of backward arcs, denoted by $B(W)$. The cost of a cycle is the algebraic sum of the costs of all arcs contained in the cycle, i.e., $c_W(\mathbf{x}) = \sum_{(i,j)\in F(W)} c_{ij}(x_{ij}) - \sum_{(i,j)\in B(W)} c_{ij}(x_{ij})$. The mean-cost of a cycle is denoted by $c_W(\mathbf{x})/|W|$. Given a cycle W, the set of nodes that have two outgoing arcs in W is denoted by $O(W)$; the set of nodes with two incoming arcs in W is denoted by $I(W)$. Given a set $Y, |Y|$ denotes the cardinality of Y.

[1] We note that Dial (2006) also mentioned ε-UE. Dial's ε-UE implies that the travel times on all paths are within ε of the cheapest path (c.f. Section 6). Our ε-optimality condition, however, is in the context of the reduced costs specified in Eq. (6), which is closely related to the min-mean cycle in the residual network.

[2] Flow operations in this paper are manipulated on a set of cycles called PAS – Paired Alternative Segments. See Definition 1.