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A partial differential equation formulation of Vickrey's bottleneck model, part I: Methodology and theoretical analysis

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ABSTRACT

This paper is concerned with the continuous-time Vickrey model, which was first introduced in Vickrey (1969). This model can be described by an *ordinary differential equation* (ODE) with a right-hand side which is discontinuous in the unknown variable. Such a formulation induces difficulties with both theoretical analysis and numerical computation. Moreover it is widely suspected that an explicit solution to this ODE does not exist. In this paper, we advance the knowledge and understanding of the continuous-time Vickrey model by reformulating it as a *partial differential equation* (PDE) and by applying a variational method to obtain an explicit solution representation. Such an explicit solution is then shown to be the strong solution to the ODE in full mathematical rigor. Our methodology also leads to the notion of *generalized Vickrey model* (GVM), which allows the flow to be a distribution, instead of an integrable function. As explained by Han et al. (in press), this feature of traffic modeling is desirable in the context of analytical *dynamic traffic assignment* (DTA). The proposed PDE formulation provides new insights into the physics of The Vickrey model, which leads to a number of modeling extensions as well as connection with first-order traffic models such as the *Lighthill–Whitham–Richards* (LWR) model. The explicit solution representation also leads to a new computational method, which will be discussed in an accompanying paper, Han et al. (in press).

1. Introduction

1.1. General background and modeling issue

The Vickrey Model (VM) is one of the most commonly employed link models in the current literature on dynamic traffic assignment. It was original presented in Vickrey (1969) and discussed subsequently, for example, by Drissi-Kaïtouni and Hameda-Benchekroun (1992), Heydecker and Addison (1996), Kuwahara and Akamatsu (1997), and Li et al. (2000). The Vickrey model is based on the assumption that the queue has negligible size and that the travel time on the link consists of a free-flow travel time plus a congestion related queuing time. A popular mathematical form of the model is an ordinary differential equation with discontinuous dependence on the state variable. Such an irregularity leads to several theoretical difficulties: a classical solution may no longer exist; and it is widely suspected that such an ODE does not admit an explicit solution representation. For a general review on ODEs with discontinuous state-dependence, the reader is referred to Filippov (1988) and Stewart (1990).

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Subsequent research based on the Vickrey model mainly focuses on two aspects: (1) modification of the original model in continuous-time, and (2) discrete-time counterpart of the model. The first line of research is primarily represented by the work of Armbruster et al. (2006a), Ban et al. (2011), and Pang et al. (2011). In the study of a *dynamic user equilibrium* (DUE) problem with a single bottleneck, Pang et al. (2011) used a *linear complementarity system* (LCS) as an approximation to the Vickrey model. Such an LCS does not admit an explicit ODE representation. Nevertheless, it was shown in Ban et al. (2011) that the LCS yields an absolutely continuous solution that satisfies the complementarity condition almost everywhere. In an attempt to approximate the LCS with an explicit ODE, Ban et al. (2011) proposed the α-model which includes a smoothing parameter $\alpha \gg 1$. The α-model has an explicit ODE representation. Moreover, it exhibits an asymptotic behavior as α approaches infinity, and the limit is precisely the LCS studied in Pang et al. (2011). The point-queue type dynamic has been employed also in the context of continuous supply chain networks. Armbruster et al. (2006a) considered an ODE similar to the α-model: that ODE also contains a smoothing parameter $0 < \varepsilon \ll 1$. Such a system, which we will subsequently call the ε-model, has been verified extensively against the *deterministic discrete event simulations* (DDES) by Armbruster et al. (2006b). Notice that all of the aforementioned continuous-time models deviate from Vickrey's original model, in exchange for the theoretical and computational convenience. In addition, the smoothing parameters introduced by the α-model and the ε-model need to be further understood in terms of their underlying physical meanings and modeling implications. A more detailed review of these models is provided by this paper, in Section 3.

So far, the Vickrey model or the point-queue type models are mainly studied in discrete time and in a numerical framework. Yet, fundamental issues regarding convergence and physical realism of the numerical solutions remain relatively less understood. Ban et al. (2011) showed that both forward and backward finite-difference schemes for the Vickrey ODE may yield negative queue lengths and/or negative flows. A few adaptations of the finite-difference algorithms that would avoid the above problems were proposed, for example, by Armbruster et al. (2006a), Ban et al. (2011), and Pang et al. (2011). However, these improved algorithms still bear certain numerical limitations. For example, the α-model by Ban et al. (2011) could potentially lead to negative queues if a forward scheme is used. Furthermore, the forward scheme of the α-model exhibits conditional stability. The ε-model by Armbruster et al. (2006a) could potentially lead to non-physical solutions in some rare circumstances, see Han et al. (2012b) for an example. One should notice that the performances of both the α-model and the ε-model are greatly influenced by the smoothing parameters chosen, which could potentially add to the complexity and uncertainty of the corresponding dynamical system.

In view of the above limitations, a mathematically rigorous investigation of the original Vickrey's model in continuous time will not only provide new insights into the physics of the model, but also plow new grounds for the analysis and computation of the Vickrey model in the context of DTA. This motivates our two-part work.

1.2. Methodology based on partial differential equations

It has not been noted before that one may analyze the Vickrey model from the approach of partial differential equations. The PDEs are playing important roles in hydrodynamic traffic flow models such as the Lighthill–Whitham–Richards model by Lighthill and Whitham (1955) and Richards (1956). A list of selected references on the LWR model is Bressan and Han (2011a,b), Claudel and Bayen (2010a,b), Daganzo (1994, 1995, 2005), Garavello and Piccoli (2006), Han et al. (2012c), and Newell (1993). The PDE formulation of the Vickrey model is motivated by the simple observation that the Vickrey model, like many hydrodynamic models, is based on conservation of cars. Such a principle manifests itself in the generic form

$$\partial_t \rho(t,x) + \partial_x f(t,x) = 0 \tag{1.1}$$

where $\rho(t,x)$ and $f(t,x)$ denotes respectively the density and flow at time t and location x. The spatial dimension in (1.1) can be naturally embedded in the Vickrey model due to the presence of a free-flow phase (although our analysis presented in this paper works even if the free-flow time is zero). The bottleneck located at the exit of a link can be modeled by an x-dependent flow capacity constraint imposed on the entire link. This requires that the flow is not only a function of density (which is the case for LWR model), but also a function of the spatial variable x. A more detailed and formal discussion based on the above observations can be found in Section 4.1.

Once a conservation law describing the point-queue dynamic of the Vickrey model is obtained, we can readily derive a Hamilton–Jacobi equation by integrating the scalar conservation law. The key component of our analysis is a variational method for the H–J equation, which is known as the Lax–Hopf formula (Bressan and Han, 2011a; Daganzo, 2005; Evans, 2010; Le Floch, 1988; Lax, 1957). The Lax–Hopf formula was originally proposed as a semi-analytic representation of the solutions to the scalar conservation law and the Hamilton–Jacobi equation. Its application to first-order traffic flow models are recently investigated by Aubin et al. (2008), Bressan and Han (2011a,b), Claudel and Bayen (2010a,b), and Daganzo (2005). The variational approach expresses the viscosity solution to the Hamilton–Jacobi equation as an optimization problem. However, note that the Lax–Hopf formula does not immediately apply to because the Hamiltonian of our derived H–J equation has a discontinuous dependence on the spatial variable x. As a consequence, the H–J equation does not admit solutions in the viscosity sense, see Evans (2010) for more details. In other words, one can only expect the solution to be defined in the distributional sense, i.e. the density $\rho(t,x)$ can contain dirac-delta which is precisely the mathematical abstraction of the "point queue". Fortunately, as we show in this paper, it is possible to apply the Lax–Hopf formula in a novel way even though the resulting density is a distribution. Technical detail of the analysis is presented in Section 4.2.