Transportation Research Part B 46 (2012) 156–174

Contents lists available at SciVerse ScienceDirect

Transportation Research Part B

journal homepage: www.elsevier.com/locate/trb

# A stochastic model of traffic flow: Theoretical foundations

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ARTICLE INFO

*Article history:*
Received 5 June 2011
Received in revised form 11 September 2011
Accepted 11 September 2011

*Keywords:*
Stochastic traffic flow
Random time headways
Counting processes
Cell transmission
Fluid limit

ABSTRACT

In a variety of applications of traffic flow, including traffic simulation, real-time estimation and prediction, one requires a probabilistic model of traffic flow. The usual approach to constructing such models involves the addition of random noise terms to deterministic equations, which could lead to negative traffic densities and mean dynamics that are inconsistent with the original deterministic dynamics. This paper offers a new stochastic model of traffic flow that addresses these issues. The source of randomness in the proposed model is the uncertainty inherent in driver gap choice, which is represented by random state dependent vehicle time headways. A wide range of time headway distributions is allowed. From the random time headways, counting processes are defined, which represent cumulative flows across cell boundaries in a discrete space and continuous time conservation framework. We show that our construction implicitly ensures non-negativity of traffic densities and that the fluid limit of the stochastic model is consistent with cell transmission model (CTM) based deterministic dynamics.



## 1. Introduction

A variety of traffic management applications today require probabilistic models of traffic flow. These include traffic simulation, estimation of traffic conditions along freeways and signalized arterials when measurement data is limited (whether in real-time or not), and applications that involve short-term traffic prediction such as online (adaptive) traffic signal control. An issue that has received little attention in the literature pertaining to the stochastic modeling of traffic flow is that of the "physical relevance" of the sample paths of stochastic processes. Any stochastic model of traffic flow, in some *mean dynamic sense*, should describe queue build-up and dissipation in a manner that is consistent with well established (deterministic) traffic flow principles, such as predictions of the model of Lighthill, Whitham, and Richards (the LWR model) (Lighthill and Whitham, 1955; Richards, 1956). In fact, traffic flow variables in the LWR model are defined as *averages* (e.g., in (Lighthill and Whitham, 1955) traffic density is defined as "the average number of vehicles on the slice of road"). From the kinetic view of traffic flow Prigogine and Herman (1971), the LWR model arises as a temporal mean dynamic. In addition, such stochastic dynamics should implicitly ensure the non-negativity of traffic densities because ad hoc treatments of negative traffic densities may reinforce the issue of inconsistency with deterministic traffic flow models.

In the literature, the usual line of attack when constructing stochastic models of traffic flow is to select a deterministic traffic flow model, which usually consists of a conservation equation and a fundamental relationship, and then add "noise" terms to the model. These include the models of Gazis and Knapp (1971) and Szeto and Gazis (1972) for purposes of traffic state estimation and control (also later in Gazis and Liu (2003)), where noise is added to the conservation equation (and the measurement equation). This is also the case in Wang and Papageorgiou (2005) and Wang et al. (2007), where noise is added to the dynamic speed equation and to a speed-density relationship (their model of traffic state includes both a conservation

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doi:10.1016/j.trb.2011.09.006

of flow equation and a speed evolution equation). Other recent examples include Boel and Mihaylova (2006) and Sumalee et al. (2011), which constitute stochastic extensions of the cell transmission model (Daganzo, 1994; Daganzo, 1995b).

In such settings, and particularly when the deterministic dynamics are nonlinear, one cannot guarantee that the resulting mean dynamics are consistent with the original deterministic dynamics. To illustrate, let $\bar{z}(t)$ denote a dynamic variable of interest (i.e., one that varies with time, denoted by $t$) and suppose it evolves according to:

$$\bar{z}(t+\Delta t) = g(\bar{z}(t), \Delta t), \tag{1}$$

a deterministic dynamic. Let $\zeta(t)$ denote white noise. A stochastic dynamic, written as:

$$z(t+\Delta t) = g(z(t), \Delta t) + \zeta(t+\Delta t) \tag{2}$$

suffers two problems. First, if $g(\cdot,\cdot)$ is non-linear (a typical feature of traffic flow models), then it is easy to see that, in general, $\mathbb{E}z(t+\Delta t) \neq \bar{z}(t+\Delta t)$,[1] despite the fact that $\mathbb{E}\zeta(t) = 0$ for each $t \geqslant 0$. The second problem is that arbitrarily adding the white noise term, $\zeta(\cdot)$, results in negative sample paths. One may suggest, to overcome this problem, that the noise terms can simply be replaced with random variables that only assume positive values (e.g., truncated noise terms); unfortunately, such random variables cannot have zero mean (unless they are identically 0) and the first problem is reinforced. To the best of the authors' knowledge, this problem has not been identified in the literature of stochastic traffic flow models and has yet to be addressed in the theory of stochastic conservation laws (see for example Holden and Risebro (1997) for a brief discussion and numerical illustration). A notable exception, in the traffic flow literature, is the Markov compartment model (MARCOM) (Davis and Kang, 1994; Kang, 1995), which models traffic flows as state-dependent Poisson processes. While the issue of non-negativity of sample paths does not arise in MARCOM, the resulting mean dynamics generally do not conform to LWR theory as this was not an objective in their study.

To overcome these problems, this paper proposes a new stochastic model of traffic flow that operates in discrete space (i.e., we break road sections into small cells) and continuous time. The use of continuous time is a consequence of random time headways taking their values in $\mathbb{R}^+$; it is also motivated by our desire to apply the proposed model to actuated traffic signals, where signal phases may vary in length. The use of discrete space is analytical tractability. The proposed dynamics can be thought of as probabilistic versions of the cell transmission model (or, more generally, Godunov scheme based dynamics), but unlike previous literature that generalizes the cell transmission model (CTM), our concern is that the CTM arise as a mean dynamic. The source of randomness in the proposed model is the uncertainty inherent in driver gap choice, which is represented by random vehicle time headways. Generally speaking, adding noise may be appropriate for systems where uncertainty is due to exogenous stochastic forcing, but this is not the case when considering conservation of traffic on a road without sources or sinks; here, a dominant source of uncertainty is driver behavior. To achieve consistency with the CTM, random elements in the model depend on traffic state; namely, the traffic density. In the proposed model, when traffic densities becomes zero, the random variables describing cumulative traffic flow across cell boundaries will degenerate (i.e., their variance becomes zero). This assumption has physical relevance. Take flow rates across a certain boundary for instance: given the traffic density is equal to zero, one knows *with certainty* that the flow rate is zero as there are no vehicles to cross the boundary; also, when traffic is at jam density, one knows with certainty that the flow rate is zero as well. In general, uncertainty in traffic state increases with increasing vehicle interactions (e.g., lane-changes and deceleration behind slower vehicles); thus the need to incorporate state dependence in traffic variables.

In the proposed model, we do not suppose any particular headway distribution; we only assume finite variances; the proposed model can, hence, accommodate a variety of headway distributions, which is desirable from an application standpoint. We also allow various flux-density relationships, but shall restrict our analysis to concave relationships and our examples to triangular relationships. To show that our model is consistent with first-order traffic flow theory, we derive a (deterministic) *fluid limit* of the proposed stochastic model, which we show to be consistent with the CTM. Estimation of system states and parameters will not be covered in the present paper, but shall be treated in a sequel along with real world examples.

This paper is organized as follows: in Section 2, we give a brief background on deterministic dynamics and how they arise as long-run averages of general kinetic models of traffic flow. We then proceed to Section 3, the main section in this paper, in which we discuss our proposed approach for modeling vehicle time headways, how this is used to construct our proposed stochastic model, ensuring non-negativity of traffic densities, and a formal derivation of the fluid limit of our model. Throughout Section 3, we give small examples to both illustrate the generality of the model and attempt to help hone the intuition behind the mathematics. In Section 4, two numerical examples are presented to demonstrate the consistency with the CTM and the evolution of probability densities of time headways in space and time as a result of the propagation of a traffic disturbance. Section 5 concludes the paper and discusses, briefly, future research directions.

## 2. Background

### 2.1. The LWR model: properties and solution

To set the stage for the remainder of this paper, we give a brief review of the Lighthill–Whitham and Richards (LWR) model of macroscopic traffic flow (Lighthill and Whitham, 1955; Richards, 1956) and its most popular solution techniques,

[1] $\mathbb{E}$ denotes the *expectation* operator.