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On the applicability and solution of bilevel optimization models in transportation science: A study on the existence, stability and computation of optimal solutions to stochastic mathematical programs with equilibrium constraints

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ABSTRACT

Bilevel optimization models, and more generally MPEC (mathematical program with equilibrium constraints) models, constitute important modelling tools in transportation science and network games, as they place the classic "what-if" analysis in a proper mathematical framework. The MPEC model is also becoming a standard for the computation of optimal design solutions, where "design" may include either or both of network infrastructure investments and various types of tolls. At the same time, it does normally not sufficiently well take into account possible uncertainties and/or perturbations in problem data (travel costs and demands), and thus may not a priori guarantee robust designs under varying conditions. We consider natural stochastic extensions to a class of MPEC traffic models which explicitly incorporate data uncertainty. In stochastic programming terminology, we consider "here-and-now" models where decisions on the design must be made before observing the uncertain parameter values and the responses of the network users, and the design is chosen to minimize the expectation of the upper-level objective function. Such a model could, for example, be used to derive a fixed link pricing scheme that provides the best revenue for a given network over a given time period, where the varying traffic conditions are described by distributions of parameters in the link travel time and OD demand functions. For a general such SMPEC network model we establish not only the existence of optimal solutions, but in particular their stability to perturbations in the probability distribution. We also provide convergence results for general algorithmic schemes based on the penalization of the equilibrium conditions or possible joint upper-level constraints, as well as for algorithms based on the discretization of the probability distribution, the latter enabling the utilization of standard MPEC algorithms. Especially the latter part utilizes relations between the traffic application of SMPEC and stochastic structural topology optimization problems.

1. Introduction and motivation

1.1. Motivation

When addressing network design and pricing problems under user equilibrium, or any kind of traffic assignment model for that matter, in the form of a hierarchical (that is, bilevel) optimization model (or, mathematical programming problem

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with equilibrium constraints, MPEC) it is almost always understood that data is certain or that uncertain data can be given some deterministic representation, such as a mean value. It is at the same time understood by all who develop or utilize traffic equilibrium or assignment models, that this assumption is rather simplistic. One might take the extreme approach that hierarchical models based on simple deterministic and static assignment models therefore are of little use, and that position would most likely not change even if the underlying static traffic model would be replaced by a dynamic model. Replacing a deterministic traffic equilibrium model by a stochastic one does not really solve the problem either, because at least the classic ones (logit and probit, and the latter's modern relatives) also produce unique equilibrium solutions in general; hence the mapping from data to solution is not stochastic even in such models.

Since uncertainty in data is prevalent in traffic assignment models, and because bilevel models provide such an attractive modelling approach, there is a need for a bilevel model that can take into account the uncertainty in data. (For discussions regarding the demand function, see, e.g., Yang et al., 1991; Flyvbjerg et al., 2005; Yang et al., 2005; Chen et al., 2006; discussions on proper travel cost/time functions are too numerous to mention, but a few basic references are Outram and Thompson, 1978; Akcelik, 1991; Uchida and Iida, 1993).

One possibility is to use a classic 'multi-load' approach, taken from the engineering sciences, in which several representative scenarios are included through an averaging in the upper-level objective function. Structural optimization is the science of producing structures that carry loads optimally. Variations in, for example, loading conditions are often taken into account such that the effectiveness of a design is defined as the average value of the effectiveness of several 'load cases'. Because of the anticipated fact that the 'real' probability model is never known, and the reported high sensitivity of solutions to stochastic structural optimization problems with respect to small changes in the probability measure (Ben-Haim and Elishakoff, 1990, pp. 20–22), many probability-free worst-case ('pessimistic') models of uncertainty have been developed as alternatives to probabilistic ones. In such worst-case models, uncertain parameters are assumed to vary in convex sets (see further below). An efficient numerical approach to solve certain convex problems of this type is known (Ben-Tal and Nemirovski, 1997), but it has considerable drawbacks, the most serious in our context being that the uncertain data must lie in some small ellipsoid around the primal data values, which of course reduces the generality of the algorithm substantially.

This paper seeks to extend the scope of bilevel traffic models to take into account data variations in the form of a stochastic bilevel model, or SMPEC. In order to do so we must consider complications that do not arise in standard bilevel traffic models: SMPEC models are infinite-dimensional, so even the existence of optimal solutions is non-trivial to establish. The main focus of our study of the SMPEC traffic model of this paper is however one that, to some degree, can be used to validate the use of MPEC models in network design applications and in particular multi-load approaches and discretizations: we wish to establish conditions under which optimal designs change continuously with the probability distribution. We say that such solutions are stable, and the model therefore is 'robust' in some sense. Establishing the robustness of the stochastic bilevel model to be developed in this paper is also an alternative to the approach taken in some papers, where robust models are produced through reformulation (see below).

This work is built on a talk at the Royal Society discussion meeting "Networks: modelling and control" held in London 24–25 September 2007, and on the corresponding discussion paper (Patriksson, 2008); the current work however is much more complete, including proofs of all the theorems stated (some of which are also new) as well as including a more thorough section on further research and a more complete bibliography, compared to the afore-mentioned discussion paper.

1.2. Stochastic programming and robust optimization

The problem of data uncertainty has been recognized as important for quite some time in the operations research community. The most well-known technique for dealing with uncertainty is stochastic programming (e.g., Van Slyke and Wets, 1969; Rockafellar and Wets, 1976; Kall and Wallace, 1994; Prékopa, 1995; Birge and Louveaux, 1997). Typically, one wants to minimize the expected cost of decisions, that must be made without the complete knowledge of data. In 'here-and-now' models decisions are made once, but in certain types of applications decisions can be made at one or several future recourse stages. In the latter case, the here-and-now decision represents the first-stage decision which is made to properly hedge against future outcomes of the uncertain data. In these types of problems one assumes that the distribution of uncertainty is known, or, at least, can be well approximated.

The term 'robust optimization' (RO) has been and is used in a variety of contexts. An important aspect in most applications is the assumption that the distribution of data is known to be confined to certain (typically bounded)'uncertainty sets'. For particular problem types, such as linear programming (e.g., Soyster, 1973; Ben-Tal and Nemirovski, 1999, 2002), convex quadratic programming (e.g., El Ghaoui and Lebret, 1997), and certain maximum-stiffness truss topology optimization problems (e.g., Ben-Tal and Nemirovski, 1997; Ben-Tal et al., 1999), and particular uncertainty sets, RO offers computationally tractable (i.e., polynomially solvable) robust versions. Because feasibility is required for every realization of the data, the robust counterparts include a semi-infinite system of constraints (whence tractability is only possible to obtain in the above types of instances). Importantly, it is at the same time a *worst-case* modelling approach.

Through the above modelling technique one imposes robustness on a model. Our main interest is to investigate the robustness of SMPEC models, particularly in transportation science and network games, in the hope that a reformulation is not necessary. As we shall see, it is in some circumstances possible to establish the robustness of an SMPEC model, but reformulations may be necessary in some cases. SMPEC models, and indeed also MPEC models, are normally not tractable; robust reformulation will therefore tend not to be tractable either.