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An improved solution algorithm for the constrained shortest path problem

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Abstract

The shortest path problem is one of the classic network problems. The objective of this problem is to identify the least cost path through a network from a pre-determined starting node to a pre-determined terminus node. It has many practical applications and can be solved optimally via efficient algorithms. Numerous modifications of the problem exist. In general, these are more difficult to solve. One of these modified versions includes an additional constraint that establishes an upper limit on the sum of some other arc cost (e.g., travel time) for the path. In this paper, a new optimal algorithm for this constrained shortest path problem is introduced. Extensive computational tests are presented which compare the algorithm to the two most commonly used algorithms to solve it. The results indicate that the new algorithm can solve optimally very large problem instances and is generally superior to the previous ones in terms of solution time and computer memory requirements. This is particularly true for the problem instances that are most difficult to solve. That is, those on large networks and/or where the additional constraint is most constraining.

Keywords: Network routing; Constrained shortest path problem; Exact algorithms

1. Introduction

The Shortest Path (SP) problem is one of the oldest and most widely used problems in network optimization (Dijkstra, 1959; Dantzig, 1960; Floyd, 1962). The objective of the SP problem is to identify the least cost path (or route) through a network from a pre-determined starting node to a pre-determined terminus node. The SP problem is one of the relatively few network optimization problems for which exact, polynomial time solution algorithms exist (e.g., see Magnanti and Wong, 1984; Evans and Minieka, 1992; Daskin, 1995).

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Various "constrained" versions of the basic SP problem exist. For example, the path may be constrained to include specific nodes, or be constrained to include a specific number of nodes (e.g., see Deo and Pang, 1984), or include nodes within a pre-specified "covering" distance of every node in the network (Current et al., 1984; Current et al., 1994). In general, the term "constrained shortest path problem" (cSP) refers to the constrained problem in Handler and Zang (1980) that includes one additional constraint that establishes an upper limit on the sum of some other arc cost (e.g., travel time) for the path. Unfortunately, the addition of such constraints to the SP problem generally results in a problem that belongs to the set of problems known as NP-hard (e.g., see Garey and Johnson, 1979).

In this article, we introduce an improved exact algorithm for the cSP as defined in Handler and Zang (1980). Although this problem is NP-hard, we demonstrate that very large problem instances (40 000 nodes and 800 000 arcs) can be solved in reasonable time. We compare this new algorithm's results and solution times to the Lagrangian relaxation and the k-shortest path approaches presented in Handler and Zang (1980). The results of these tests indicate that the proposed algorithm can solve large problems in reasonable time and has advantages over the other methods in terms of solution time and computer memory requirements.

2. Mathematical formulation of the cSP problem

Consider a directed graph, $G = (N, A)$ where $N = \{1, 2, \ldots, n\}$ represents the set of nodes and $A = \{(i,j)\colon i, j \in N, i \neq j\}$ represents the set of m directed arcs. Two, non-negative weights c_{ij} and t_{ij} are associated with each arc (i,j). These weights may represent, for example, the length (or cost) and the travel time associated with the respective arc. Label the origin node, 1, and the destination node, n.

Given these definitions, the cSP problem may be formulated as a binary integer program as follows:

$$\min \quad \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_j x_{ij} - \sum_k x_{ki} = \begin{cases} 1 & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i = 2, \ldots, n-1, \\ -1 & \text{if} \quad i = n, \end{cases} \tag{2}$$

$$\sum_{(i,j)\in A} t_{ij}x_{ij} \leqslant T \tag{3}$$

$$x_{ij} \in \{0, 1\}, (i,j) \in A, \tag{4}$$

where x_{ij} are binary variables associated with each arc (i,j). If arc (i,j) is included in the optimal path, then $x_{ij} = 1$; otherwise $x_{ij} = 0$. The parameter, T, represents the maximum value allowed for the sum of the t_{ij} arc weights. For example, T could represent the maximum allowable time to transverse the path. The shortest path problem is formulated by (1), (2), and (4). As noted before, this problem can be solved optimally in polynomial time. Unfortunately the addition of constraint (3) results in a problem that belongs to the set on NP-hard problems (Garey and Johnson, 1979).

3. Existing solution algorithms for the cSP problem

Handler and Zang (1980) proposed two methods to solve the cSP problem. One requires solving a k-Shortest Path (kSP) problem and the other is based upon Lagrangian relaxation. Other approaches (e.g., Johsch, 1966, which is based on dynamic programming), have been proposed. However, the Handler and Zang Lagrangian relaxation-based method is generally considered the most efficient.

3.1. Algorithm 1: k-Shortest path method

This algorithm was introduced in Handler and Zang (1980). It identifies the optimal solution to the cSP by solving a k-Shortest Path (kSP) problem. The objective of the kSP is to identify k paths from the origin node to