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Transportation Research Part B 41 (2007) 540–553

TRANSPORTATION RESEARCH PART B

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Retaining desirable properties in discretising a travel-time model

Malachy Carey *, Y.E. Ge

School of Management and Economics, Queen's University, 25 University Square, Belfast, Northern Ireland BT7 1NN, United Kingdom

Received 12 January 2003; accepted 20 September 2006

Abstract

A recent paper introduced a new whole-link travel time model and showed that it has various desirable properties, including a first-in-first-out (FIFO) property, causality and consistency with the usual static model when flows are constant. The model is formulated as a continuous-time first-order differential equation, which does not have a general analytical solution but can be solved (approximately) numerically by forward or backward discrete-time differencing methods. Here we show that if the step sizes are not arbitrarily small then the solutions obtained by the usual differencing methods do not always preserve FIFO. In view of that, we introduce a new differencing method and prove that it always preserves FIFO and the other desirable properties exhibited by the continuous-time model. In numerical examples we illustrate how the new discrete-time differencing model eliminates FIFO violations, illustrate convergence of a solution process for the new model, and illustrate how various inflow patterns affect FIFO under the old and new differencing methods.

Keywords: Link travel time; Dynamic traffic assignment; First-in-first-out property; FIFO

1. Introduction

For traffic networks where the flows and travel times are varying over time, it is well known that if the link travel times are modelled as a function of the current inflow and/or outflow of the link, this can cause unrealistic behaviour of the travel times, including FIFO violations. To overcome this, in a recent paper Carey et al. (2003) introduced a new whole-link model. They show that the above problems are avoided if, instead of treating the link travel time as a function of the state of the link at time of entry, we treat it as a function of an estimate of the flow in the immediate neighbourhood of the vehicle, averaged over the time the vehicle spends traversing the link. This average flow rate can be approximated by an average of the inflow rate when the vehicle is entering the link and the outflow rate when it is exiting from the link. They formulate the model as a continuous-time first-order differential equation, and show that it has various desirable properties, including satisfying a first-in-first-out (FIFO) condition. The model does not have a general analytical solution, but can be solved numerically (approximated) using various discrete-time forward or backward differencing methods for solving first-order differential equations. However, in the present paper we show that the standard

* Corresponding author. Tel.: +44 2890 245133; fax: +44 2890 975156.
E-mail addresses: m.carey@qub.ac.uk (M. Carey), y.ge@qub.ac.uk (Y.E. Ge).

0191-2615/$ - see front matter
doi:10.1016/j.trb.2006.09.003

forward or backward differencing methods yield solutions that can violate FIFO, though the FIFO violations are reduced as the discretisation is refined. In view of this, we consider other solution schemes and develop an alternative differencing method that always ensures FIFO without requiring that the time steps in the discretisation be small. The method consists of applying the backward differencing method while moving *forward* in time, or applying the forward differencing method while moving *backwards* in time. We show that either of these methods always preserves FIFO and other properties of the continuous-time model.

The new solution method and model ensures FIFO even if large time steps Δt are used, which is important for various reasons. First, larger time steps reduce the amount of computation. Second, it is shown in Carey and Ge (2003) that the approximation to the LWR traffic flow model (Lighthill and Whitham, 1955; Richards, 1956) is improved if the length of the time step is near to (but not greater than) the time taken to traverse the link. A third reason is as follows. The differencing scheme yields a discrete-time version of the continuous-time model. This discrete-time version can be used to model the link travel times in discrete-time or multi-period network models for dynamic traffic assignment (DTA). In that case, the time steps in the differencing scheme may be used as the period lengths in the multi-period models for DTA, and there are various reasons for not making the latter period lengths in DTA models arbitrarily small.

The discrete-time solution methods developed and presented in this paper have been used to solve numerical examples in some previous papers by the authors, including Carey (1999, 2001) and Carey and Ge (2003, 2004), but neither the methods nor their properties nor their effect on FIFO have been set out or discussed in those papers.

So that this paper may be self-contained, Section 2 summarises the model introduced in Carey et al. (2003). Section 3 introduces standard discrete-time backward and forward differencing methods to solve the continuous-time model, shows that these can violate FIFO, introduces new differencing methods and shows that these preserve FIFO and other properties of the continuous-time model. Section 3 also presents some simple examples in which analytical solutions can be found. Numerical experiments are presented in Section 4 to illustrate FIFO violation arising from the use of the usual differencing methods, illustrate the convergence of an iterative scheme designed to solve the proposed differencing method, and illustrate the effects of various inflow profiles on FIFO under the old and new differencing schemes.

2. A continuous-time whole-link travel-time model satisfying FIFO

The continuous-time model of Carey et al. (2003) can be set out as follows. For a vehicle entering a link at time t, let the time $\tau(t)$ that it takes to traverse the link be a function $h(w(t))$ of a flow rate $w(t)$, where $w(t)$ is a weighted average of the inflow rate when the vehicle is entering the link and the outflow rate when the vehicle is exiting from the link. When the vehicle is entering the link at time t let the inflow be $u(t)$ and when it is exiting at time $t + \tau(t)$ let the outflow be $v(t + \tau(t))$. Hence the weighted average of these can be written as

$$w(t) = \beta u(t) + (1 - \beta)v(t + \tau(t)), \tag{1}$$

where β is a constant and $0 \leqslant \beta < 1$. Letting the link travel time $\tau(t)$ be a function of this weighted average flow gives

$$\tau(t) = h(w(t)) = h(\beta u(t) + (1 - \beta)v(t + \tau(t))), \tag{2}$$

where $h(\cdot)$ is a given, continuous, nondecreasing or strictly increasing function. (Strictly increasing is needed when we later use the inverse $h^{-1}(\cdot)$.) The weighted average (1) can be thought of as an approximation to the "average" of the flow rates in the immediate neighbourhood of the vehicle over the time that the vehicle travels along the link. The travel-time function (2) is then simply the well-known static travel-time model $\tau = h(u)$ with the constant flow u replaced by the weighted average flow rate (1).

The functional form and parameters for the travel time function $h(\cdot)$ in this paper, as in Carey et al. (2003), are assumed to be the same as in the widely used 'static' travel time functions that are used for example in static traffic assignment network models. In the latter, the link travel time is written as a function of the link flow and both are assumed constant over time. Since such travel time functions have been in use, discussed and estimated for decades (e.g. see Transportation Research Board, 2000), it is not necessary to discuss them or their estimation further here. Also, it is of course well-known that the travel time functions, flow density