Transportation Research Part B 45 (2011) 176–207

Contents lists available at ScienceDirect

Transportation Research Part B

journal homepage: www.elsevier.com/locate/trb

Approximate network loading and dual-time-scale dynamic user equilibrium

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ARTICLE INFO

Article history:
Received 10 November 2009
Received in revised form 10 May 2010
Accepted 10 May 2010

Keywords:
Dynamic user equilibrium
Differential variational inequalities
Differential algebraic equations
Dual-time-scale
Fixed-point algorithm in Hilbert space

ABSTRACT

In this paper we present a dual-time-scale formulation of dynamic user equilibrium (DUE) with demand evolution. Our formulation belongs to the problem class that Pang and Stewart (2008) refer to as differential variational inequalities. It combines the within-day time scale for which route and departure time choices fluctuate in continuous time with the day-to-day time scale for which demand evolves in discrete time steps. Our formulation is consistent with the often told story that drivers adjust their travel demands at the end of every day based on their congestion experience during one or more previous days. We show that analysis of the within-day assignment model is tremendously simplified by expressing dynamic user equilibrium as a differential variational inequality. We also show there is a class of day-to-day demand growth models that allow the dual-time-scale formulation to be decomposed by time-stepping to yield a sequence of continuous time, single-day, dynamic user equilibrium problems. To solve the single-day DUE problems arising during time-stepping, it is necessary to repeatedly solve a dynamic network loading problem. We observe that the network loading phase of DUE computation generally constitutes a differential algebraic equation (DAE) system, and we show that the DAE system for network loading based on the link delay model (LDM) of Friesz et al. (1993) may be approximated by a system of ordinary differential equations (ODEs). That system of ODEs, as we demonstrate, may be efficiently solved using traditional numerical methods for such problems. To compute an actual dynamic user equilibrium, we introduce a continuous time fixed-point algorithm and prove its convergence for effective path delay operators that allow a limited type of nonmonotone path delay. We show that our DUE algorithm is compatible with network loading based on the LDM and the cell transmission model (CTM) due to Daganzo (1995). We provide a numerical example based on the much studied Sioux Falls network.

1. Introductory remarks

Dynamic traffic assignment (DTA) is usually viewed as the positive (descriptive) modeling of time varying flows on vehicular networks consistent with established traffic flow. This paper is concerned with a specific type of dynamic traffic assignment known as *continuous time dynamic user equilibrium* (DUE) for which unit travel cost, including early and late arrival penalties, is identical for those route and departure time choices selected by travelers between a given origin–destination

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0191-2615/$ - see front matter
doi:10.1016/j.trb.2010.05.003

pair. We study the special case for which travel demand is constant within each given day of interest, although it evolves from day to day. This special case is the simplest plausible circumstance under which a discrete-time, day-to-day model of demand learning may be coupled to a continuous-time, within-day DUE model. In presenting such a dual-time-scale theory, we employ demand evolution dynamics motivated by evolutionary game theory. No doubt more complicated dual-time-scale models can and will be proposed; nonetheless, it is fitting that, as the first such model reported in the dynamic traffic assignment literature, our formulation is particularly easy to understand and to solve by a combination of decomposition via time-stepping and fixed point iterations in a function space.[3]

In Sections 2 and 3, we quickly review some already well known material concerning the history of dynamic user equilibrium modeling and computation. In Section 4, we present our dual-time-scale formulation. In the same section, we also show that any continuous time within-day variational inequality formulation of DUE is naturally re-expressible as a differential variational inequality. We go on to show that such a differentiable variational inequality is easily analyzed using elementary optimal control theory. That analysis, elegant in its simplicity and conciseness, is important from a pedagogical point of view and should make the theory of continuous time DUE accessible to a wider body of scholars. Section 6 present a fixed point algorithm for within-day DUE and its convergence for a limited class of non-monotone delay operators. The same section also presents an algorithm for calculating the day-to-day evolution of travel demand consistent with intra-day dynamic user equilibria. Section 7 discusses dynamic network loading from the point of view of differential algebraic equations and gives a detailed procedure for DNL when the link delay model of Friesz et al. (1993) is employed. Section 8 presents the results of several numerical examples.

2. Current state of DUE scholarly inquiry

Numerous scholarly teams working independently around the globe have slowly made advances in modeling and computing dynamic use equilibria. In fact, DUE modeling and computation have now reached a point where substantial agreement exists regarding the general content of a mathematical model of dynamic user equilibrium, the desired standards of performance for algorithms that compute DUE flow patterns, and critical unanswered research questions. To document this emerging consensus, in Section 3 we review some of the most significant analytical dynamic user equilibrium models that have been proposed, along with associated algorithms for their solution. That review stresses DUE modeling perspectives that are widely held and widely employed for DUE computational research. Those perspectives must be assessed in the light of available and emerging mathematical and algorithmic tools; when that is done, the following observations may be made:

Observation I. Among analytical DUE models, there are no dual-time-scale models recognizing tactical routing and departure time decisions are made in continuous time (the within-day time scale) while demand evolves in discrete time (the day-to-day time scale) and that the two time scales are coupled, although there is considerable agreement that this dichotomy of time scales is apropos.

Observation II. We may distinguish two essential aspects of modeling dynamic user equilibrium for the within-day time scale: (i) dynamic network loading and (ii) simultaneous route and departure time equilibria, where dynamic network loading (DNL) subsumes the modeling of delay, flow evolution (arc dynamics) and flow propagation (enforcement of traffic laws during flow evolution).

Observation III. Simultaneous route and departure time choice are integral to the definition of a dynamic user equilibrium and have to date been mainly expressed as variational inequalities, quasi-variational inequalities or complementarity problems, either in discrete time or continuous time. However, the emerging literature on abstract *differential variational inequalities* has not been well exploited for either modeling or computing simultaneous route and departure time equilibria.

Obsevation IV. Little agreement exists regarding an appropriate mathematical formulation of network loading. Furthermore, the emerging literature on *differential algebraic equations*, despite its focus on problem structures like those encountered in network loading, has not been exploited.

Observation V. Fully general path delay operators may fail to be monotonic and/or differentiable. Rigorously convergent algorithms for determining path departure rates that constitute a user equilibrium for such general path delay operators have not been available.

In this paper we make contributions related to each of the above observations. Specifically, we accomplish the following:

1. the formulation of a dual-time-scale model of dynamic user equilibrium that endogenizes day-to-day evolution of travel demand;
2. the expression of a version of simultaneous path and departure time equilibrium as a *differential* variational inequality that subsumes several of the key models we review, simplifies the analysis of equilibrium conditions, and provides direct access to the growing literature on differential variational inequalities;

[3] Evolutionary game theory is an appealing foundation for modeling day-to-day demand evolution, as it is principally concerned with dynamic learning processes. Excellent presentations of evolutionary game theory are provided by Hofbauer and Sigmund (1998) and Samuelson (1998).