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Development of the convolutions of truncated normal random variables with three different quality characteristics in engineering applications

Russell Krenek, Jinho Cha, Byung Rae Cho *

Advanced Quality Engineering Laboratory, Department of Industrial Engineering, Clemson University, Clemson, SC 29634, USA

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A B S T R A C T

In real-world situations, specifications are implemented to screen out nonconforming products as a part of screening inspections, which result in a truncated distribution for conforming products. Understanding these truncated probability density functions is paramount to the overall manufacturing industry, as more accurate evaluations of a process output will lead to a greater understanding of the process itself and associated costs. Furthermore, convolutions of truncated random variables play an important role in a multistage manufacturing system, where a screening inspection is performed at each stage. While the convolutions of normal distributions have been well established, the convolutions of truncated normal distributions have neither been understood clearly, nor have theoretical foundations been thoroughly explored in the literature, despite their practical importance. The mathematical framework and approximations using the error function for the convolutions of truncated normal random variables with three different types of quality characteristics are presented here. The convolutions established in this paper should enhance the accuracy and precision of real-world production processes particularly where components are required for assembling into the final product.

1. Introduction

There are three kinds of quality characteristics, which are smaller-the-better (S), nominal-the-best (N), and larger-the-better (L) types. Fig. 1 shows four different types of a truncation which are often implemented in the quality engineering field. They are (1) a two-sided, asymmetric left and right truncated distribution at the lower specification limit (LSL or x_l) and the upper specification limit (USL or x_u) for an N-type characteristic, (2) a two-sided, asymmetric left and right truncated distribution at the LSL and USL for an N-type quality characteristic, (3) a one-sided left truncated distribution at the LSL for an L-type quality characteristic, and (4) a one-sided right truncated distribution at the USL for an S-type quality characteristic. Shown in Fig. 1, the dotted and solid lines represent normal distributions before inspection and truncated normal distributions after inspection, respectively.

Understanding truncated random variables and their roles in screening inspection is paramount to modern industry, as this type of inspection arises in many engineering applications. First, final products are often subjected to screening, and only conforming products are distributed to the customer, while the rejected products, which do not meet the specification requirements, are scrapped or reworked. This screening inspection of the products results in a truncated distribution which represents the conforming products delivered to the customer, as shown in Fig. 1. This concept has been well advanced in the literature. See, for example, Barr and Sherrill (1999), Kim and Takayama (2003), Jawitz (2004), Khasawneh, Bowling, Kaewkuekool, and Cho (2005a, 2005b), Makarov, Loutan, Ma, and de Mello (2009), Cha and Cho (2014), Cha, Cho, and Sharp (2014). Second, another potential application of screening inspection can be found in a multistage assembly production process, as shown in Fig. 2, where only conforming items are passed on to the next stage. Examples of multistage processes are numerous. Typical systems of telecommunication, banking, and healthcare consist of multistage processes. It is noted that a product part or service transferring from one stage to the next stage in a multistage process may introduce extra variations that do not occur in a generic single-stage process. An added advantage of screening inspection in a multistage process is the ability of reducing the extra variations by screening noncomforming items in each stage. This can lead to a greater understanding of production costs and more accurate understanding of process outputs.

It is important to note that these multistage screening inspection processes require the sum of truncated random variables, also

* Corresponding author. Tel.: +1 864 656 1874; fax: +1 864 656 0795.
E-mail addresses: rkrenek@g.clemson.edu (R. Krenek), jcha@g.clemson.edu (J. Cha), bcho@clemson.edu (B.R. Cho).

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0360-8352/

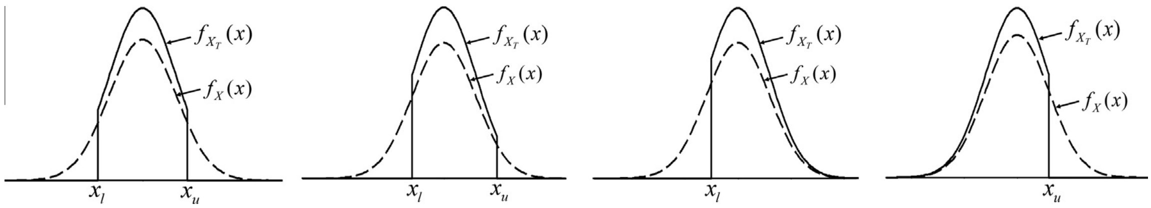

Fig. 1. Four different types of a truncated normal distribution.

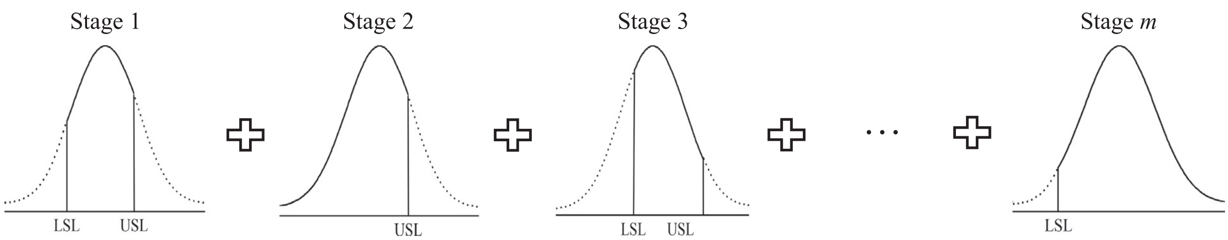

Fig. 2. Screening inspection in a multistage assembly process.

called the convolution of truncated distributions. These convolutions resulting from multiple truncations have practical importance in mechanical assembly design, statistical tolerance design, gap analysis, and other quality engineering areas; however, the mathematical framework of the convolutions associated with truncated distributions have not been well established in the literature.

This paper focuses on the development of the convolutions resulting from two and three times truncated normal random variables, which are referred to as the convolutions of truncated normal distributions on double and triple truncations. Table 1 shows the current body of knowledge associated with the convolutions of truncated normal distributions under three types of a quality characteristic, where X_{T_i} represents the ith truncated normal random variable, and $\mathrm{STN_N}$, $\mathrm{ATN_N}$, $\mathrm{TN_L}$, and $\mathrm{TN_S}$ denote symmetric and asymmetric truncated normal random variable for N-type characteristics, and left and right truncated normal random variables for L-type and S-type characteristics, respectively. As shown in Table 1, there is no known literature that explores anything other than these convolutions of the same type of a truncated distribution. Those unexplored convolutions will be developed in this paper. A general form of the convolutions of more than three truncated normal random variables is also developed. As such, this paper offers several novel contributions to the literature. Specifically, finding the probability density functions associated with the convolutions of truncated normal distributions on double and triple truncations can substantially help understand process yields and reduce variation.

Table 1
The current body of knowledge associated with the convolutions of truncated normal distributions under three types of a quality characteristic.

X_{T_1}	X_{T_2}	Status on a double truncation	X_{T_1}	X_{T_2}	X_{T_3}	Status on a triple truncation
$\mathrm{STN_N}$	$\mathrm{STN_N}$	Available[1]	$\mathrm{STN_N}$	$\mathrm{STN_N}$	$\mathrm{STN_N}$	Available[1]
$\mathrm{TN_S}$	$\mathrm{TN_S}$	Available[2]	$\mathrm{TN_S}$	$\mathrm{TN_S}$	$\mathrm{TN_S}$	Available[2]
$\mathrm{TN_L}$	$\mathrm{TN_L}$	Available[3]	$\mathrm{TN_L}$	$\mathrm{TN_L}$	$\mathrm{TN_L}$	Available[3]
$\mathrm{ATN_N}$	$\mathrm{ATN_N}$	Unexplored	$\mathrm{ATN_N}$	$\mathrm{ATN_N}$	$\mathrm{ATN_N}$	Unexplored
$\mathrm{STN_N}$	$\mathrm{ATN_N}$	Unexplored	$\mathrm{STN_N}$	$\mathrm{ATN_N}$	$\mathrm{STN_N}$	Unexplored
					$\mathrm{TN_L}$	
					$\mathrm{TN_S}$	
	$\mathrm{TN_L}$			$\mathrm{TN_L}$	$\mathrm{STN_N}$	Unexplored
	$\mathrm{TN_S}$			$\mathrm{TN_S}$		
	$\mathrm{TN_L}$			$\mathrm{TN_L}$	$\mathrm{TN_S}$	
$\mathrm{ATN_N}$	$\mathrm{STN_N}$	Unexplored	$\mathrm{ATN_N}$	$\mathrm{STN_N}$	$\mathrm{ATN_N}$	Unexplored
	$\mathrm{TN_L}$			$\mathrm{TN_L}$		
	$\mathrm{TN_S}$			$\mathrm{TN_S}$		
	$\mathrm{TN_L}$			$\mathrm{TN_L}$	$\mathrm{TN_S}$	
$\mathrm{TN_L}$	$\mathrm{STN_N}$	Unexplored	$\mathrm{TN_L}$	$\mathrm{STN_N}$	$\mathrm{TN_L}$	Unexplored
	$\mathrm{ATN_N}$			$\mathrm{ATN_N}$		
	$\mathrm{TN_S}$			$\mathrm{TN_S}$		
$\mathrm{TN_S}$	$\mathrm{STN_N}$	Unexplored	$\mathrm{TN_S}$	$\mathrm{STN_N}$	$\mathrm{TN_S}$	Unexplored
	$\mathrm{ATN_N}$			$\mathrm{ATN_N}$		
	$\mathrm{TN_L}$			$\mathrm{TN_L}$		

[1] Birnbaum and Andrews (1949).
[2] Francis (1946).
[3] Kratuengarn (1973).

2. Literature review

The probability density functions and cumulative density functions of the sums of identical and independently distributed truncated normal random variables were first established by Francis (1946). Birnbaum and Andrews (1949) then discussed sums of symmetrically truncated normal random variables and Aggarwal and Guttman (1960) introduced hypothesis tests to truncated normal random variables. Lipow, Mantel, and Wilkinson (1964) further developed the probability functions of convolutions of standard normal random variables and extended this concept to left truncated normal random variables.

As for a single truncation, Field, Harder, and Harrison (2004) studied truncated distributions associated with the measurement of traffic from different locations in relation to high-performance Ethernet and data security, respectively. More recently, Flecher, Allard, and Naveau (2010) examined an expression of the moments based on a truncated skew normal distribution. Kuo and Tsai (2011) applied the Monte Carlo method to obtain the densities of the sums of truncated normal random variables with 1,000,000 samples. Examples of applying truncated distributions to semiconductor processes include (Kim & May, 1999), who introduced a via formation procedure on multichip integrated circuit modules with dielectric layers of the module being made of photosensitive benzocyclobutene. The via formation process involves several sequential unique stages: spin coating, soft baking, exposing, developing, curing and plasma descuming. Examples of truncated distributions in the aircraft industry include the C-17 and F/A-18 E/F wing assemblies (Inman, Carbrey, Calawa, & Hartman, 1996).

Another common application area of truncated normal distributions can be found in statistical tolerance and gap analysis. The application of statistical techniques to tolerances used in