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Uncertain multilevel programming: Algorithm and applications

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ABSTRACT

Multilevel programming is used to model a decentralized planning problem with multiple decision makers in a hierarchical system. This paper aims at providing an uncertain multilevel programming model that is a type of multilevel programming involving uncertain variables. Besides, a genetic algorithm is employed to solve the model. As an illustration, the uncertain multilevel programming model is applied to a product control problem.

1. Introduction

Multilevel programming was first proposed by Bracken and McGill (1973) to model a decentralized noncooperative decision system with one leader and multiple followers of equal status in 1973. It finds many applications in daily life such as strategic-force planning (Bracken & McGill, 1974), resource allocation (Aiyoshi & Shimizu, 1981), and water regulation (Anandalingam & Apprey, 1991). In 1990, Ben-Ayed and Blair (1990) showed that multilevel programming is an NP-hard problem. In order to solve the model numerically, many algorithms have been proposed such as extreme point algorithm (Candler & Towersley, 1982), *k*th best algorithm (Bialas & Karwan, 1984), branch and bound algorithm (Bard & Falk, 1982), descent method (Savard & Gauvin, 1994), and intelligent algorithm (Liu, 1998).

However, in many cases, the parameters in the multilevel programming are indeterminate. Multilevel programming involving random variable was first proposed by Patriksson and Wynter (1999) in 1999. In addition, Gao, Liu, and Gen (2004) proposed some new stochastic multilevel programming models in 2004. Multilevel programming involving fuzzy set was first proposed by Lai (1996) in 1996, and then developed by Shih, Lai, and Lee (1996), and Lee (2001). Especially, Gao and Liu (2005) proposed a new fuzzy multilevel programming model, and defined a Stackelberg–Nash equilibrium.

As we know, a premise of applying probability theory is that the obtained probability distribution is close enough to the true frequency. In order to get it, we should have enough samples. But due to economical or technical difficulties, we sometimes have no samples. In this case, we have to invite some domain experts to evaluate the belief degree that each event happens. However, a lot of surveys showed that human beings usually estimate a much wider range of values than the object actually takes (Liu, 2015). This conservatism of human beings makes the belief degrees deviate far from the frequency. As a result, the belief degree cannot be treated as probability distribution, otherwise some counterintuitive phenomena may happen (Liu, 2012). In order to deal with the belief degree mathematically, an uncertainty theory was founded by Liu (2007) in 2007, and refined by Liu (2010) in 2010. A concept of uncertain variable is used to model uncertain quantity, and belief degree is regarded as its uncertainty distribution. As a type of mathematical programming involving uncertain variables, uncertain programming was founded by Liu (2009) in 2009. So far, uncertain programming has been applied to many fields such as project scheduling, vehicle routing, facility location, and system design.

In this paper, we will propose a framework of uncertain multilevel programming. The rest of the paper is organized as follows. In Section 2, we review some concepts and theorems in uncertainty theory. In Section 3, we introduce the basic form of uncertain programming. The uncertain multilevel programming is proposed in Section 4, and its equivalent model is obtained and a genetic algorithm to solve the model is introduced in Section 5. In order to illustrate the efficiency of the algorithm, an example of production control is proposed in Section 6. At last, some remarks are made in Section 7.

2. Preliminary

In order to model human's belief degree, an uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010) in

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2010 as a branch of axiomatic mathematics. Nowadays, it has been widely applied to mathematical programming, and has brought out a branch of uncertain programming (Liu, 2009) which is a spectrum of mathematical programming involving uncertain variables. So far, uncertain programming has been applied to shortest path problem (Gao, 2011), facility location problem (Gao, 2012; Wen, Qin, & Kang, 2014), employment contract model (Mu, Lan, & Tang, 2013), inventory problem (Qin & Kar, 2013), spanning tree (Zhang, Wang, & Zhou, 2013), and so on.

The basic concept of uncertainty theory is uncertain measure, which is used to indicate the belief degree of each event.

Definition 1 Liu, 2007. Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra on Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms,

Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3: (Subadditivity Axiom) For every sequence of $\{\Lambda_i\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leqslant \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Besides, a product axiom was given by Liu (2009) for the operation of uncertain variables in 2009.

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Uncertain variable is used to represent quantities in uncertainty. Essentially, it is a measurable function on an uncertainty space.

Definition 2 Liu, 2007. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain variable ξ is a measurable function from Γ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 3 Liu, 2009. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

In order to describe an uncertain variable in practice, a concept of uncertainty distribution is defined below.

Definition 4 Liu, 2007. The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leqslant x\}$$

for any real number x.

If an uncertainty distribution has an inverse function, then the inverse function is called an inverse uncertainty distribution. In this case, the uncertainty distribution is called regular. Inverse uncertainty distributions play an important role in the operation of uncertain variables. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Liu (2010) showed that if the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution

$$\Psi^{-1}(r) = f(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r)).$$

The expected value of an uncertain variable is an average of the uncertain variable in the sense of uncertain measure.

Definition 5 Liu, 2007. The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty}\mathcal{M}\{\xi \geqslant x\}dx - \int_{-\infty}^{0}\mathcal{M}\{\xi \leqslant x\}dx$$

provided that at least one of the two integrals is finite.

Assuming that ξ has an uncertainty distribution Φ, Liu (2007) proved

$$E[\xi] = \int_0^{+\infty}(1 - \Phi(x))dx - \int_{-\infty}^{0}\Phi(x)dx.$$

Furthermore, Liu and Ha (2010) proved that the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r))dr.$$

Here, the function f and the uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are as aforementioned.

3. Uncertain programming – basic form

Assume that $\boldsymbol{x}$ is a decision vector, and ξ is an uncertain vector. Since an uncertain objective function $f(\boldsymbol{x}, \xi)$ cannot be directly maximized, we may maximize its expected value, i.e.,

$$\max_{\boldsymbol{x}} E[f(\boldsymbol{x}, \xi)].$$

In addition, since the uncertain constraints $g_j(\boldsymbol{x}, \xi) \leqslant 0$, $j = 1, 2, \ldots, p$ do not define a crisp feasible set, it is naturally desired that the uncertain constraints hold with confidence levels $\alpha_1, \alpha_2, \ldots, \alpha_p$. Then we have a set of chance constraints,

$$\mathcal{M}\{g_j(\boldsymbol{x}, \xi) \leqslant 0\} \geqslant \alpha_j, \quad j = 1, 2, \ldots, p.$$

In order to obtain a decision with maximum expected objective value subject to a set of chance constraints, Liu (2009) proposed the following uncertain programming model,

$$\begin{cases} \max\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \xi)] \\ \text{subject to :} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \xi) \leqslant 0\} \geqslant \alpha_j, \quad j = 1, 2, \ldots, p. \end{cases} \tag{1}$$

Definition 6. A vector $\boldsymbol{x}$ is called a feasible solution to the uncertain programming model (1) if

$$\mathcal{M}\{g_j(\boldsymbol{x}, \xi) \leqslant 0\} \geqslant \alpha_j$$

for $j = 1, 2, \ldots, p$.