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A note on critical-set and lifted surrogate inequalities for cardinality-constrained linear programs ☆

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ABSTRACT

We study the polyhedral approach to the cardinality-constrained linear programming problem (CCLP). First, we generalize Bienstock's *critical-set inequalities*. We find necessary and sufficient conditions for the generalized inequalities to define facets of the convex hull of CCLP's feasible set. Then, we show how to derive *lifted surrogate* cutting planes on-the-fly. We test the use of both families of inequalities on branch-and-cut to solve difficult instances of CCLP to proven optimality. Our computational results indicate that the use of the inequalities can reduce the time required to solve CCLP by branch-and-cut considerably.

1. Introduction

Let m, n and K be positive integers, $M = \{1, \ldots, m\}$, and $N = \{1, \ldots, n\}$. The *cardinality-constrained linear programming problem* (CCLP) is

$$\begin{aligned} \text{maximize} \quad & \sum_{j \in N} c_j x_j \\ \text{subject to} \quad & \sum_{j \in N} a_{ij} x_j \geq b_i \quad i \in M && (1) \\ & x_j \geq 0 \quad j \in N && (2) \\ & x_j \leq 1 \quad j \in N && (3) \\ & \text{at most } K \text{ variables } x_j \text{ can be nonzero.} && (4) \end{aligned}$$

Constraint (4), which we call *cardinality*, arises for example in finance (Perold, 1984; Qiu, Ahmed, Dey, & Wolsey, 2014) and petroleum engineering (Vasantharajan & Cullick, 1997); see also Boyd (2012), Kellerer, Pferschy, and Pisinger (2004), Buglieri, Ehrgott, Hamacher, and Maffioli (2006). de Farias and Nemhauser (2003) showed that CCLP is NP-hard even when $m = 1$.

We assume that:

Assumption 1. $a_1 \geq \cdots \geq a_n$.

Assumption 2. $K \leq n - 1$.

Assumption 3. $\sum_{j=1}^{K} a_j \geq b$.

Assumption 4. $b \geq 0$ *and* $a_j \geq 0$, $\forall j \in N$.

Assumption 1 is without loss of generality (WLOG). When Assumption 2 does not hold, (4) is redundant, so it is also WLOG. Given Assumptions 1, 3 guarantees feasibility. Assumption 4 carries loss of generality. However, it is satisfied in important applications, e.g. portfolio selection (Bienstock, 1996).

Let $S = \{x \in \mathbb{R}^n \colon (2)\text{–}(5) \text{ hold}\}$, where

$$\sum_{j \in N} a_j x_j \geq b \qquad (5)$$

is one of the inequalities (1), and let $P = conv(S)$. Here, we study two families of inequalities valid for S and their use within branch-and-cut to solve CCLP to proven optimality.

The polyhedron P, under Assumption 4, was studied by Bienstock (1996), who in particular introduced *critical-set inequalities* for P. On the other hand, P, under the different assumption that $b < 0$ and $a_j < 0$, $\forall j \in N$ was studied by de Farias and Nemhauser (2003). The case where the variables are binary was considered by Stephan (2010) and Zeng and Richard (2011).

Here we give two new inequalities for P. Our first family of inequalities generalizes Bienstock's critical-set inequalities. Our second family of inequalities is obtained by lifting *surrogate cardinality constraints*, i.e., inequalities of the type

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$$\sum_{j \in N'} x_j \leq K, \tag{6}$$

where $N' \subset N$.

Given a set of indices $A \subseteq N$, we denote the minimum and maximum elements of A as $\min(A)$ and $\max(A)$, respectively. We let $\overline{A} = N \setminus A$ and $\widehat{A} = A \cup \{j \in N : j > \max(A)\}$. For a given nonnegative integer t, we let A_t be the set consisting of the t smallest elements of A (if $t = 0, A_t = \emptyset$, and if $t > |A|$, $A_t = A$). Finally, we define $\sum_{i \in \emptyset} a_i = 0$.

In Section 2 we study the *trivial* facets of P. In Section 3 we generalize Bienstock's critical-set inequalities. In Section 4 we consider the problem of lifting inequalities valid for P and we discuss lifted surrogate constraints. In Section 5 we explain our separation heuristics and lifting algorithm. In Section 6 we report the results of our computational experience with instances of CCLP. In Section 7 we present conclusions and directions for further research.

2. Trivial inequalities

We call *trivial* the valid inequalities that are "easily" inferred by the definition of P. In this section we provide necessary and sufficient conditions for them to be facet-defining. We also give necessary and sufficient conditions for P to be full-dimensional. The proofs of the propositions of this section are simple, and therefore omitted (see Kozyreff, 2014 for the proofs).

Proposition 1. *The polytope P is full-dimensional iff the following conditions hold:*

$$\sum_{j=1}^{K} a_j > b \tag{7}$$

and

$$\sum_{j=1}^{K-1} a_j + a_n \geq b. \tag{8}$$

Because the inequality description of a polyhedron is much simpler when it is full-dimensional, we assume that:

Assumption 5. Conditions (7) and (8) hold.

Proposition 2. *Inequality (5) defines a facet of P.*

Proposition 3. *Inequality (2) defines a facet of P, $\forall j > K$. It is facet-defining for $j \leq K$ iff $\sum_{i=1}^{K+1} a_i - a_j > b$ and $\sum_{i=1}^{K} a_i - a_j + a_n \geq b$.*

If (2) is not facet-defining for some $j \leq K$, then, by the previous proposition, at least one of the two conditions must fail. If $\sum_{i=1}^{K+1} a_i - a_j \leq b$, then (5) forces every x_i with $i \leq j$ to be nonnegative. If $\sum_{i=1}^{K} a_i - a_j + a_n < b$, then $x_n > 0$ forces every x_i with $i \leq j$ to be positive.

Proposition 4. *Inequality (3) defines a facet of P $\forall j < K$. It is facet-defining for $K \leq j \leq n-1$ iff $\sum_{i=1}^{K-1} a_i + a_j > b$ and $\sum_{i=1}^{K-2} a_i + a_j + a_n \geq b$. Finally, it is facet-defining for $j = n$ iff $\sum_{i=1}^{K-1} a_i + a_n > b$ and $\sum_{i=1}^{K-2} a_i + a_{n-1} + a_n \geq b$.*

If (3) is not facet-defining for some $j \geq K$, then there exists a set $C \subset N$, $|C| \geq 2$, such that $j \in C$ and $\sum_{i \in C} x_i \leq 1$ is valid for P. We will return to this in the next section, when we study *critical-set inequalities.*

Proposition 5. The inequality

$$\sum_{j \in N} x_j \leq K \tag{9}$$

is valid for P. It defines a facet of P iff $\sum_{j=2}^{K+1} a_j \geq b$.

Example 1. Let $S = \{x \in [0,1]^8 : 11x_1 + 10x_2 + 9x_3 + 8x_4 + 5x_5 + 2x_6 + 2x_7 + 1x_8 \geq 21$, and at most 3 variables x_j can be positive$\}$. Then P is full dimensional, $x_2 \geq 0, \ldots, x_8 \geq 0$ define facets of P ($x_1 \geq 0$ does not), $x_1 \leq 1, \ldots, x_3 \leq 1$ define facets of P ($x_4 \leq 1, \ldots, x_8 \leq 1$ do not), and $\sum_{j=1}^{8} x_j \leq 3$ is facet-defining for P.

Example 2. Let $S = \{x \in [0,1]^8 : 11x_1 + 10x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 + 2x_7 + 1x_8 \geq 23$, and at most 4 variables x_j can be positive$\}$. Then P is full dimensional, both $x_j \geq 0$ and $x_j \leq 1$ define facets of P for all $j \in \{1, \ldots, 8\}$, and $\sum_{j=1}^{8} x_j \leq 4$ is facet-defining for P.

3. Generalized critical-set inequalities

We now generalize Bienstock's critical-set inequalities (Bienstock, 1996), and we give necessary and sufficient conditions for them to define facets of P.

Definition 1. (Bienstock (1996)) A set $C \subset N$ is *critical* if $\forall J \subset C$ with $|J| = K$, $\sum_{j \in J} a_j < b$.

Definition 2. Let $d \in \{2, \ldots, K\}$. A set $C \subset N$, with $|C| \geq d$, is *d-critical* if

$$\sum_{j \in \overline{C}_{K-d}} a_j + \sum_{j \in C_d} a_j < b.$$

A d-critical set is *maximal* if $\forall j \in \overline{C}, C \cup \{j\}$ is not d-critical.

If C is d-critical, then no more than $d-1$ variables with index in C can be positive. Thus,

$$\sum_{j \in C} x_j \leq d - 1 \tag{10}$$

is valid for P. Note that if C is d-critical, then so is $\widehat{C}$. We call (10) a *generalized critical-set inequality*. We now give necessary and sufficient conditions for generalized critical-set inequalities to define facets.

Proposition 6. *Let $C \subset N$ be a maximal d-critical set, with $d \in \{2, \ldots, K\}$, and suppose that $\max(\overline{C}_{K-d+1}) < \min(C)$. Then (10) is facet-defining for P iff the following conditions hold:*

(i) $\sum_{j \in \overline{C}_{K-d+1}} a_j + \sum_{i \in C_d} a_j - a_{\min(C)} \geq b$
(ii) $\sum_{j \in \overline{C}_{K-d+1}} a_j + \sum_{i \in C_{d-2}} a_j + a_{\max(C)} \geq b$
(iii) $\sum_{j \in \overline{C}_{K-d+1}} a_j + \sum_{i \in C_{d-1}} a_j > b.$

Proof. Suppose that (i)–(iii) hold. Since C is maximal, $\max(C) = n$, and (ii) implies that $\sum_{j \in \overline{C}_{K-d}} a_j + a_i + \sum_{j \in C_{d-1}} a_j \geq b$, for all $i \in \overline{C} \setminus \overline{C}_{K-d+1}$.

Let $y \in (0,1)$ such that $\sum_{j \in \overline{C}_{K-d}} a_j + a_{\max(\overline{C}_{K-d+1})} y + \sum_{j \in C_{d-1}} a_j > b$. The following n points belong to P, are linearly independent, and satisfy (10) at equality:

For $i \in C_d : x_j^i = 1\ \forall j \in \overline{C}_{K-d+1}, x_j^i = 1\ \forall j \in C_d \setminus \{i\}$, and $x_j^i = 0$ otherwise; for $i \in C \setminus C_d : x_j^i = 1\ \forall j \in \overline{C}_{K-d+1}, x_j^i = 1\ \forall j \in C_{d-2}, x_i^i = 1$, and $x_j^i = 0$ otherwise; for $i \in \overline{C}_{K-d+1} : x_j^i = 1\ \forall j \in \overline{C}_{K-d+1} \setminus \{i\}, x_i^i = 1\ \forall j \in C_{d-1}, x_i^i = y$, and $x_j^i = 0$ otherwise; for $i \in \overline{C} \setminus \overline{C}_{K-d+1} : x_j^i = 1\ \forall j \in \overline{C}_{K-d}, x_j^i = 1\ \forall j \in C_{d-1}, x_i^i = 1$, and $x_j^i = 0$ otherwise.

Suppose now that (10) is facet-defining for P. If (i) does not hold, then $x_{\min(C)} = 1$ for every feasible point satisfying (10) at equality. If (ii) does not hold, then $x_n = 0$ for every feasible point satisfying (10) at equality. If (iii) does not hold, then every point satisfying (10) at equality also satisfies (5) at equality. □