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A queue with working breakdowns

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ABSTRACT

In this paper, we consider a new class of queueing models with working breakdowns. The system may become defective at any point of time when it is in operation. However, when the system is defective, instead of stopping service completely, the service continues at a slower rate. Using the probability generating function, we give the joint distribution of the server state and the number of customers in the system in steady state. We also derive the necessary and sufficient condition for the existence of the steady state. We study the waiting time distribution of our model. Finally, some performance measures and numerical examples are presented.

1. Introduction

One important fact that has been overlooked is that perfectly reliable servers are virtually nonexistent. In fact, the servers may well be subject to lengthy and unpredictable breakdowns while serving a customer. For example, in manufacturing systems the machine may breakdown due to machine or job related problems. This results in a period of unavailable time until the servers are repaired. Such a system with repairable server has been studied as a queueing model and reliability model by many authors.

Cao and Cheng (1982) have considered an $M/G/1$ queueing model with a repairable server. Li, Shi, and Chao (1997) have considered an $M/G/1$ queueing system with server breakdowns and Bernoulli vacations and have discussed the mean number of customers in the system. Krishna Kumar, Arivuadainambi, and Vijayakumar (2002) have considered an $M/G/1/1$ queueing model with unreliable server and no waiting capacity. The server is assumed to provide two types of services regular and optional with the provision of server breakdown. Gray, Wang, and Scott (2004) have considered a queueing model with multiple types of server breakdowns. Wang (2004) studied an $M/G/1$ queue with a second optional service and server breakdowns. Several authors have considered queueing models with N policy and server breakdowns.

From the mid-20th century, computer technology and networks have been utilized in many areas including telecommunication, flexible manufacturing, e-commerce and supply-chain systems. These systems usually are operated under random environments and congestion is often caused by the mismatch between variable service capacity and random service demand. Therefore, it is becoming increasingly important to study the performance characteristics of such computer systems. There are situations in the real world, where the breakdown of a server may not stop the service of a customer completely.

For example, the presence of a virus in the system may slow down the performance of the computer system. The computer system may still be able to perform various chores but at a considerably slower rate. However, in all the models considered so far of queueing systems with server breakdowns, the underlying assumption has been that a server breakdown disrupts the service completely in the system. Another example is provided by the machine replacement problem. A machine may suddenly breakdown when it is in operation. It is immediately replaced by another standby machine which may work at a slower rate. As soon as the broken down machine is repaired it is put back into service. Here the failure of the machine does not stop the work completely. However, in all the models considered so far of queueing systems with server breakdowns, the underlying assumption has been that a server breakdown disrupts the service completely in the system. Motivated by this factor, we have therefore considered in this paper a new class of queueing systems with working breakdowns.

This concept of working breakdowns is different from the concept of working vacations, first formulated by Servi and Finn (2002) and then considered by Liu, Xu, and Tian (2002), Jain and Agrawal (2007), Tian, Zhao, and Wang (2008), Tian, Li, and Zhang (2009), Wu and Takagi (2006), Li, Liu, and Tian (2010), Goswami and Selvaraju (2010), Jain and Jain (2010), Zhang and Hou (2010), Yang, Wang, and Wu (2010), Jain and Upadhyaya (2011), Li and Tian (2011). A working vacation is taken only when the server

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has completed the service of all the customers present in the system. However, a working breakdown, can occur at any time point at which the server is busy with the service of a customer in the system. Sridharan and Jayashree (1996) have considered a queueing system with both partial and total failure of the server and a finite capacity. However, they have assumed that the server breaks down even when the server is idle. In our paper, we have assumed that the server fails only in the operational state. We have considered an infinite capacity queueing system. We have derived the necessary and sufficient condition for the stability of the system and the waiting time distribution of our model.

The remaining part of the paper is structured as follows. The next section gives a description of the queueing model. In Section 3, we study the sufficient condition for the stability of our model. In Section 4, we analyze the steady-state distribution of our queueing system. Waiting time distribution of our model is discussed in Section 5. Some performance measures are discussed in Section 6. In Section 7, we give some numerical results. In Section 8, we give a few concluding remarks and some suggestions for future research.

2. The *M*/*M*/1 model

We consider a single server queueing system. The arrivals are in accordance with a Poisson process with a rate λ. It is assumed that when the server is busy, the server is subject to random breakdowns . The service times in the normal state are assumed to be exponentially distributed with a parameter μ. When the server breaks down, the service rate decreases. The service times after a breakdown of the server are assumed to be exponentially distributed with a parameter $\mu_1 < \mu$. The time until the random breakdown of the server is assumed to be exponentially distributed with a parameter α. The repair time of the server is assumed to be exponentially distributed with a parameter β. The interarrival times, the service times, the repair times and the failure times are assumed to be mutually independent of each other. Let $C(t)$ be the server state at time t. Then

$$C(t) = \begin{cases} 1, & \text{the server is in normal state} \\ 2, & \text{the server is defective} \end{cases}$$

Let $N(t)$ be the number of customers in the system at time t. Then $\{(C(t),N(t)), t \geqslant 0\}$ is a continuous time Markov chain. Let

$$P_{i,n}(t) = Prob\{C(t) = i, N(t) = n\}, \ i = 1,2. \text{ and } n \geqslant 0$$

3. A sufficient condition for the stability of the system

Let us investigate the stability condition of our model. Let $\{t_r; r \in Z_+\}$ be the sequence of epochs at which either an arrival occurs, a service completion occurs, working breakdown occurs or system returns to normal state after repair. $L_r = (C(t_r),N(t_r))$ be the system state just after the time t_r. Thus the sequence of random vectors $\{L_r; r \geqslant 1\}$ forms a Markov chain with state space $E = \{(i,n): i = 1,2 \text{ and } n \geqslant 0\}$.

Theorem 1. *The inequality* $\rho = \frac{\lambda(\alpha+\beta)}{\alpha\mu_1+\beta\mu} < 1$ *is a sufficient condition for the system to be stable.*

Proof 1. It is easy to see that $\{L_r; r \geqslant 1\}$ is an irreducible and aperiodic Markov chain. To prove positive recurrence, we may use Foster's criterion, which states that an irreducible and aperiodic Markov chain χ_i with state space S, a sufficient condition for the ergodicity is the existence of a nonnegative function $f(s)$, $s \in S$ and $\epsilon > 0$ such that the mean drift

$$x_s = E[f(\chi_{i+1}) - f(\chi_i) | \chi_i = s]$$

is finite for all $s \in S$ and $x_s \leqslant -\epsilon$ for all $s \in S$ except perhaps a finite number. In our case, we choose

$$f(s) = \begin{cases} n + B, & \text{if } s = (1,n) \\ n + B', & \text{if } s = (2,n) \end{cases}$$

where B and B' are constants which will be determined later.

For $n \geqslant 1$,

$$x_s = \begin{cases} \frac{(B'-B)\alpha+\lambda-\mu}{\alpha+\lambda+\mu}, & \text{if } s = (1,n) \\ \frac{(B-B')\beta+\lambda-\mu_1}{\lambda+\beta+\mu_1}, & \text{if } s = (2,n) \end{cases}$$

The Markov chain $\{L_n; n \geqslant 1\}$ is ergodic if the constants B and B' are chosen to satisfy the following conditions

$$(B' - B)\alpha + \lambda - \mu < 0$$
$$(B - B')\beta + \lambda - \mu_1 < 0$$

Therefore $\frac{\lambda-\mu_1}{\beta} < B' - B < \frac{\mu-\lambda}{\alpha}$. This is possible only if $\frac{\lambda-\mu_1}{\beta} < \frac{\mu-\lambda}{\alpha}$. This condition means that $\frac{\lambda(\alpha+\beta)}{\alpha\mu_1+\beta\mu} < 1$, i.e., $\rho < 1$. Thus we have that $\rho < 1$ is a sufficient condition for the ergodicity of $\{L_n; n \geqslant 1\}$. Therefore, our system attains the steady state if $\rho < 1$.

4. Steady state analysis

In this section, we examine the steady state behavior of the system. We therefore assume that $\rho < 1$. The steady state equations governing the model are

$$\lambda P_{1,0} = \mu P_{1,1} + \beta P_{2,0} \tag{4.1}$$
$$\{\lambda + \alpha + \mu\} P_{1,n} = \lambda P_{1,n-1} + \mu P_{1,n+1} + \beta P_{2,n}, \ n \geqslant 1 \tag{4.2}$$
$$\{\lambda + \beta\} P_{2,0} = \mu_1 P_{2,1} \tag{4.3}$$
$$\{\lambda + \beta + \mu_1\} P_{2,n} = \lambda P_{2,n-1} + \mu_1 P_{2,n+1} + \alpha P_{1,n}, \ n \geqslant 1 \tag{4.4}$$

Define partial probability generating functions as follows

$$P_1(z) = \sum_{n=1}^{\infty} P_{1,n} z^n \tag{4.5}$$

$$P_2(z) = \sum_{n=0}^{\infty} P_{2,n} z^n \tag{4.6}$$

Multiplying (4.1) and (4.2) by appropriate powers of z and summing over $n \geqslant 0$, we obtain

$$\{\lambda z(1-z) + \alpha z + \mu(z-1)\} P_1(z) - \beta z P_2(z) = \lambda z(z-1) P_{1,0} \tag{4.7}$$

Multiplying (4.3) and (4.4) by appropriate powers of z and summing over $n \geqslant 0$, we obtain,

$$-\alpha z P_1(z) + \{\lambda z(1-z) + \beta z + \mu_1(z-1)\} P_2(z) = \mu_1(z-1) P_{2,0} \tag{4.8}$$

Solving the Eqs. (4.7) and (4.8), we get

$$P_1(z) = \frac{[\lambda z(\lambda z - \mu_1)(1-z) + \lambda\beta z^2] P_{1,0} + \beta\mu_1 z P_{2,0}}{[(\mu - \lambda z)(1-z) - \alpha z][(\mu_1 - \lambda z)(1-z) - \beta z] - \alpha\beta z^2}(z-1) \tag{4.9}$$

$$P_2(z) = \frac{\alpha z^2 \lambda P_{1,0} + [(1-z)(\lambda z\mu_1 - \mu\mu_1) + \alpha z\mu_1] P_{2,0}}{[(\mu - \lambda z)(1-z) - \alpha z][(\mu_1 - \lambda z)(1-z) - \beta z] - \alpha\beta z^2}(z-1) \tag{4.10}$$

Now,

$$\text{denominator of } P_1(z) = [(\mu - \lambda z)(1-z) - \alpha z][(\mu_1 - \lambda z) \times (1-z) - \beta z] - \alpha\beta z^2$$

$$= (1-z)\phi(z)$$

where $\phi(z) = (\mu - \lambda z)(\mu_1 - \lambda z)(1-z) - \alpha z(\mu_1 - \lambda z) - \beta z(\mu - \lambda z)$