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An application of a merit function for solving convex programming problems

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ABSTRACT

This paper presents a gradient neural network model for solving convex nonlinear programming (CNP) problems. The main idea is to convert the CNP problem into an equivalent unconstrained minimization problem with objective energy function. A gradient model is then defined directly using the derivatives of the energy function. It is also shown that the proposed neural network is stable in the sense of Lyapunov and can converge to an exact optimal solution of the original problem. It is also found that a larger scaling factor leads to a better convergence rate of the trajectory. The validity and transient behavior of the neural network are demonstrated by using various examples.

1. Introduction

Constrained nonlinear optimization has many applications in scientific and engineering areas, such as signal and image processing, manufacturing, optimal control, and pattern recognition (Agrawal & Fabien, 1999; Avriel, 1976; Bazaraa, Sherali, & Shetty, 2006; Bertsekas, 1989; Boyd & Vandenberghe, 2004; Fletcher, 1981). Over the past years, a variety of numerical algorithms have been developed for solving constrained optimization problems, such as the simplex methods for linear programming (Rao, 2009), active set methods (Nocedal & Wright, 2006), and interior point methods (Bazaraa et al., 2006). However, traditional algorithms for digital computers may not be efficient and cannot satisfy real-time requirement such as in signal processing, robotics, function approximation and automatic control. One promising and powerful method to solve the optimization problems in real time is to employ artificial neural networks based on circuit implementation. The essence of neural network approach for mathematical programming problems is to establish an energy function (nonnegative) and a dynamic system which is a representation of an artificial neural network. The dynamic system is normally in the form of first order ordinary differential equations. It is expected that for an initial point, the dynamic system will approach its static state (or equilibrium point) which corresponds the solution of the underlying optimization problem. An important requirement is that the energy function decreases monotonically as the dynamic system approaches an equilibrium point. Because of the dynamic nature of optimization and the potential of electronic implementation, neural networks can be implemented physically by designated hardware such as application-specific integrated circuits, where the optimization procedure is truly done in parallel. Therefore, the neural network approach can solve optimization problems in running times that are orders of magnitude much faster than conventional optimization algorithms executed on general-purpose digital computers. It is of great interest to develop some neural network models that could provide a real-time online solution.

The neural network for solving mathematical programming problems was first proposed by Tank and Hopfield (1986). Kennedy and Chua (1988) proposed an improved model which employs both gradient method and penalty function method for solving nonlinear programming problems. To avoid penalty parameters, Rodriguez-Vazquez, Dominguez-Castro, Rueda, Huertas, and Sanchez-Sinencio (1990) proposed switched-capacitor neural network for solving a class of optimization problems. Based on dual and projection methods Gafini and Bertsekas (1984), Marcotte (1991), Kinderlehrer and Stampcchia (1980), Ding and Huang (2008), Gao and Liao (2010), Hu and Wang (2007), Jiang, Zhao, and Shen (2009), Maa and Shanblatt (1992), Nazemi (2011), Nazemi and Omidi (2012), Nazemi and Omidi (2013), Xia and Wang (2000), Xia and Wang (2004a), Xue and Bian (2007), Xue and Bian (2009) and Wu, Shi, Qin, Tao, and He (2010), presented

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several neural networks for solving variational inequality, convex quadratic programming, degenerate convex quadratic, degenerate quadratic minimax and interval quadratic programming problems. The projection neural networks were also developed for solving CNP problems by Effati, Ghomashi, and Nazemi (2007), Gao (2004), Leung, Chen, and Gao (2003), Liang and Wang (2000), Malek, Hosseinipour-Mahani, and Ezazipour (2010), and Xia and Wang (2004b). Recently, neural networks based on the merit functions have been designed for linear and quadratic programming, and for nonlinear complementarity problems (see Effati & Nazemi, 2006; Chen, Ko, & Pan, 2010). It is shown that these neural networks are Lyapunov stable, asymptotically stable, and exponentially stable. It is noticeable that some of the proposed neural network models have better performances than the others in theory or implementation consisting of complexity, stability and convergence.

Motivated by the above discussions, in this paper, we proposed a novel neural network for solving CNP problems based on the Karush–Kuhn–Tucker (KKT) optimality conditions and the Fischer–Burmeister (FB) merit function. This neural network is reliable and simple in structure. The proposed neural network is also proved to be globally stable in the sense of Lyapunov and can obtain an exact optimal solution of the original optimization problem.

This paper is organized as follows. In Section 2, an NCP-function is used to reformulate the CNP problem as an unconstrained minimization problem. In Section 3, a gradient neural network is constructed to solve the CNP problem and the stability properties of the proposed neural network are investigated. Some simulation results are discussed to evaluate the effectiveness of the proposed neural network in Section 4. Finally, Section 5 concludes this paper.

2. Problem statement

Consider the following CNP problem:

$$\text{minimize } f(x), \tag{1}$$

$$\text{subject to } g(x) \leqslant 0, \tag{2}$$

$$h(x) = 0, \tag{3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))^T$ is an m-dimensional vector-valued continuous function of n variables, the functions f, g_1, …, g_m are assumed to be convex and twice differentiable, $h(x) = Ax - b$, $A \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$.

Assumption 2.1. Throughout this paper, we assume the following notations as:

(a) The problem (1)–(3) has a unique optimal solution and satisfies the Slater condition (see Avriel, 1976, p. 97), i.e., there exists a $x^0 \in \mathbb{R}^n$ such that

$$g(x^0) < 0, \ Ax^0 - b = 0.$$

(b) The gradient $\{\nabla g_k(x) | k = 1, \ldots, m\} \cup \{\nabla \ h_p(x) | p = 1, \ldots, l\}$ are linear independent.

(c) $\left(\nabla^2 f(x) + \sum_{k=1}^m u_k \nabla^2 g_k(x)\right)$ is positive definite matrix on the null space of the gradients $\{\nabla h_p(x) | p = 1, \ldots, l\}$.

It is well known (see (Bazaraa et al., 2006);) that a triple $(x^{*T}, u^{*T}, v^{*T})^T \in \mathbb{R}^{n+m+l}$ is an optimal solution of (1)–(3) if and only if $(x^{*T}, u^{*T}, v^{*T})^T$ satisfies the following KKT system

$$\begin{cases} u^* \geqslant 0, \ g(x^*) \leqslant 0, \ u^{*T} g(x^*) = 0, \\ \nabla f(x^*) + \nabla h(x^*)^T v^* + \nabla g(x^*)^T u^* = 0, \\ h(x^*) = 0. \end{cases} \tag{4}$$

x^* is called a KKT point of (1)–(3) and a pair $(u^{*T}, v^{*T})^T$ is called the Lagrangian multiplier vector corresponding to x^*. Moreover, if f and g_k, $k = 1, \ldots, m$ are all convex, then x^* is an optimal solution of (1)–(3), if and only if x^* is a KKT point of (1)–(3).

For the convenience of later discussions, it is necessary to introduce a few notations, definitions and two lemmas. In what follows, $\|\cdot\|$ denotes l^2-norm of $\mathbb{R}^n$, ^{T} denotes the transpose and $x = (x_1, x_2, \ldots, x_n)^T$. If a differentiable function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$, then $\nabla \mathcal{F} \in \mathbb{R}^n$ stands for its gradient. For any differentiable mapping $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla \mathcal{F} = [\nabla \mathcal{F}_1(x), \ldots, \nabla \mathcal{F}_m(x)] \in \mathbb{R}^{n\times m}$, denotes the transposed Jacobian of $\mathcal{F}$ at x.

Definition 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open neighborhood of $\bar{x}$. A continuously differentiable function $\zeta : \mathbb{R}^n \to \mathbb{R}$ is said to be a Lyapunov function at the state $\bar{x}$ (over the set Ω) for a system $x' = \mathcal{F}(x)$, if

$$\begin{cases} \zeta(\bar{x}) = 0, \ \zeta(x) > 0, \ \forall x \in \Omega \setminus \{\bar{x}\}, \\ \frac{d\zeta(x(t))}{dt} = [\nabla \zeta(x(t))]^T \mathcal{F}(x(t)) \leqslant 0, \ \forall x \in \Omega. \end{cases}$$

Lemma 2.3 Miller and Michel (1982).

(a) *An isolated equilibrium point x^* of a system $x' = \mathcal{F}(x)$ is Lyapunov stable if there exists a Lyapunov function over some neighborhood Ω^* of x^*.*

(b) *An isolated equilibrium point x^* of a system $x' = \mathcal{F}(x)$ is asymptotically stable if there is a Lyapunov function over some neighborhood Ω^* of x^* such that $\frac{d\zeta(x(t))}{dt} < 0, \ \forall x \in \Omega^* \setminus \{x^*\}$.*

Definition 2.4. Let $x(t)$ be a solution trajectory of a system $x' = \mathcal{F}(x)$, and let X^* denotes the set of equilibrium points of this equation. The solution trajectory of the system is said to be globally convergent to the set X^*, if $x(t)$ satisfies

$$\lim_{t\to\infty} \text{dist}(x(t), X^*) = 0,$$

where $\text{dist}(x(t), X^*) = \inf_{y\in X^*} \|x - y\|$. In particular, if the set X^* has only one point x^*, then $\lim_{t\to\infty} x(t) = x^*$, and the system is said to be globally asymptotically stable at x^* if the system is also stable at x^* in the sense of Lyapunov.

Lemma 2.5. *If A is an $n \times n$ non singular matrix, then the homogeneous system $AX = 0$ has only the trivial solution $X = 0$.*

3. Reformulation and a gradient model

We can establish the relationship between the solution to problem (1)–(3) and the solution to an equivalent unconstrained minimization problem via a merit function (see Hu, Huang, & Chen, 2009; Chen et al., 2010). A merit function is a function whose global minimizers coincide with the solutions of the NCP. The class of NCP-functions defined below is used to construct a merit function.

Definition 3.1. A function $\phi : \mathbb{R}^2 \to \mathbb{R}$, is called an NCP-function if it satisfies

$$\phi(a, b) = 0 \Longleftrightarrow a \geqslant 0, \ b \geqslant 0, \ ab = 0.$$

A popular NCP-function is the FB function, which is strongly semismooth (Pan & Chen, 2010; Sun & Sun, 2005) and is defined as

$$\phi_{\text{FB}}(a, b) = (a + b) - \sqrt{a^2 + b^2}.$$

The FB merit function $\psi_{\text{FB}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ can be obtained by taking the square of ϕ_{FB}, i.e.,