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Pathological behaviors of fisher confidence bounds for Weibull distribution ☆

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ABSTRACT

The Weibull distribution is widely used in reliability engineering. To estimate its parameters and associated reliability indices, the maximum likelihood (ML) approach is often employed, and the associated Fisher information matrix is used to obtain the confidence bounds on the reliability indices that are of interest. The estimates and the confidence bounds usually behave similarly in terms of monotonic and asymptotic properties. However, the confidence bounds may behave differently under certain circumstances. As a result, the Fisher matrix approach may not always be preferred in obtaining the desired confidence bounds. This paper provides some properties of Fisher confidence bounds for the Weibull distribution. These properties can be used as guidelines when implementing the ML approach and Fisher information matrix to analyze failure time data and plan life tests.

1. Introduction

The maximum likelihood (ML) method is a popular statistical tool for estimating the parameters of a failure time distribution. It is able to fit various types of data and has many useful statistical properties such as consistency, asymptotical efficiency and transformation-invariance. These properties make it the most widely used statistical inference method to handle complete failure time data and Type I, Type II or interval censored data in life data analysis (Meeker & Escobar, 1998).

In addition to obtaining those point estimates of interest, such as specific functions associated with the failure time distribution, confidence bounds are often required in order to quantify the uncertainty of these estimates. Many statistical approaches have been developed to achieve such confidence bounds (see Meeker & Escobar, 1998, 1982). Specifically, two methods are often utilized when the ML approach is employed. One method is based on the likelihood ratio using the χ^2 distribution, and another uses the Fisher information matrix and Taylor approximation. Wiel and Meeker (1990) compare these two methods by conducting extensive simulations. The results show that the likelihood ratio method may give more accurate bounds when the sample size is relatively small. However, this method requires many iterations, and a constrained maximum likelihood estimate (MLE) needs to be obtained in each iteration. It is not uncommon that the bounds may not be found because the roots of the associated likelihood ratio function do not exist. Cheng and Iles (1983) and Cheng and Iles (1988) propose a method to calculate confidence bounds for the cumulative distribution function (*cdf*) for the location-scale distribution family based on the Fisher information matrix. By assuming that the MLEs of the distribution parameters asymptotically follow the multivariate normal distribution, a joint range of these estimates is constructed based on the χ^2 distribution. From this range, the bounds of the *cdf* can be calculated. Although the bounds have been provided to analyze complete failure time data, they cannot be applied directly to other data types or for other reliability metrics such as conditional reliability and failure rate.

An alternative way of calculating confidence bounds is to use the Fisher information matrix along with *s*-normal approximation. Compared to the likelihood ratio and Cheng and Iles' methods, this method can be easily implemented and is not limited to certain data types. This approach plays an important role in life data analysis and accelerated life testing (ALT) data analysis. One of its most successful applications is the optimum design of test plans studied by Nelson and Kielpinski (1976), Nelson and Meeker (1978), Miller and Nelson (1983), Meeker (1984), Bai, Kim, and Lee (1989), Khamis and Higgins (1996), Pascual and Montepiedra (2003), Pascual and Montepiedra (2005), Yang (2005), and Tang and Xu (2005). Due to its popularity and ease to use, it has been included in many reliability analysis software packages.

The Fisher bounds are based on the large sample *s*-normal approximation. However if this condition is not satisfied, the resulting bounds would not be accurate. The accuracy of these bounds on parameters and failure time percentiles for the Weibull distribution has been studied by Wiel and Meeker (1990) via simulation. Analytical analysis of reliability bounds is provided by

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Nomenclature

ML	maximum likelihood
MLE	maximum likelihood estimation (or estimator)
ALT	accelerated life test
CL	confidence level
pdf	probability density function
cdf	cumulative distribution function
$f(t)$	*pdf* of the Weibull distribution
$F(t)$	*cdf* of the Weibull distribution
$L(\cdot)$	likelihood function
K_f	number of groups of failure time data
N_i	number of failures in the *i*th failure time group
t_i	failure time of the *i*th failure time group
K_S	number of groups of censoring time data
N'_i	number of censoring times in the *i*th censoring time group
t'_i	time of the *i*th censoring time group
K_I	number of groups of interval failure time data
N''_i	number of failures in the *i*th interval
t''_{iL}	the start time of the *i*th interval
t''_{iR}	the end time of the *i*th interval
$\beta,\ \hat{\beta}$	shape parameter of the Weibull distribution and its MLE
$\eta,\ \hat{\eta}$	scale parameter of the Weibull distribution and its MLE
$\Sigma_{\hat{\underline{\Theta}}}$	variance–covariance matrix of MLEs of parameter vector $\underline{\Theta}$
T	percentile of a failure time distribution
$R(T)$	reliability function
$\lambda(T)$	failure rate of a failure time distribution

Zhao, Pan, Aron, and Mettas (2006) for ALT involving time-varying stresses.

In many applications of Fisher bounds, it is often found that the bounds have abnormal behaviors. For instance, the reliability function is a decreasing function with time while its bounds may increase at the beginning and then start decreasing dramatically. Such pathological behaviors have confused many engineers and researchers in practice. In this paper, theoretical insights into such abnormal phenomena are provided. In particular, Fisher confidence bounds on failure time percentiles, reliability function and failure rate for the Weibull distribution are investigated. Conditions under which the Fisher information bounds can be applied are discussed in detail. Moreover, detailed study is performed to investigate the effects of MLEs, their asymptotic variances and the assigned confidence levels on the trend of these bounds. The understanding of these effects is very important, especially for failure time data analysis and planning reliability tests.

2. Preliminaries

2.1. Model and parameter estimation

The probability density function (*pdf*) $f(t)$ and *cdf* $F(t)$ of the Weibull distribution are:

$$f(t)=\frac{\beta}{\eta^{\beta}}t^{\beta-1}e^{-\left(\frac{t}{\eta}\right)^{\beta}}, \tag{2.1}$$

$$F(t)=1-e^{-\left(\frac{t}{\eta}\right)^{\beta}}, \tag{2.2}$$

where β is the shape parameter and η is the scale parameter. Let $\underline{\Theta}$ be the vector $[\beta,\eta]$. The general formulation for the likelihood function, including exact failure time, censoring time and interval censored time data, can be written as:

$$L(\beta,\eta)=\prod_{i=1}^{K_f}f(t_i)^{N_i}\prod_{i=1}^{K_S}\left(1-F(t'_i)\right)^{N'_i}\prod_{i=1}^{K_I}\left(F(t''_{iR})-F(t''_{iL})\right)^{N''_i}. \tag{2.3}$$

Taking the natural logarithm of Eq. (2.3) yields the log-likelihood function:

$$\ln L(\beta,\eta)=\Lambda=\sum_{i=1}^{K_f}N_i\ln f(t_i)+\sum_{i=1}^{K_S}N'_i\ln\left(1-F(t'_i)\right)+\sum_{i=1}^{K_I}N''_i\ln\left(F(t''_{iR})-F(t''_{iL})\right). \tag{2.4}$$

The MLE of β and η can be found by solving $\partial\Lambda/\partial\beta=0$ and $\partial\Lambda/\partial\eta=0$ simultaneously. This paper studies the cases where solutions for β and η and the Fisher information matrix discussed in Section 2.2 exist. For more discussions on the MLE, readers are referred to Meeker and Escobar (1998), Nelson (1982) and Elsayed (1996).

2.2. The fisher information matrix and inference

The Fisher information matrix associated with the Weibull distribution is given by:

$$F_0=\begin{bmatrix}E_0\left[-\frac{\partial^2\Lambda}{\partial\beta^2}\right] & E_0\left[-\frac{\partial^2\Lambda}{\partial\beta\partial\eta}\right]\\ E_0\left[-\frac{\partial^2\Lambda}{\partial\eta\partial\beta}\right] & E_0\left[-\frac{\partial^2\Lambda}{\partial\eta^2}\right]\end{bmatrix}. \tag{2.5}$$

The subscript 0 indicates that the quantity is evaluated at the true values of the parameters. When failure data are available, one can compute the "local" information matrix by:

$$F=\begin{bmatrix}-\frac{\partial^2\Lambda}{\partial\beta^2} & -\frac{\partial^2\Lambda}{\partial\beta\partial\eta}\\ -\frac{\partial^2\Lambda}{\partial\eta\partial\beta} & -\frac{\partial^2\Lambda}{\partial\eta^2}\end{bmatrix}_{\beta=\hat{\beta},\eta=\hat{\eta}}. \tag{2.6}$$

Eq. (2.6) calculates the Fisher information using the estimated parameters. By taking the inverse of F, the local estimate of the variance–covariance matrix can be obtained as:

$$\Sigma_{\hat{\underline{\Theta}}}=\begin{bmatrix}Var(\hat{\beta}) & Cov(\hat{\beta},\hat{\eta})\\ Cov(\hat{\beta},\hat{\eta}) & Var(\hat{\eta})\end{bmatrix}=\begin{bmatrix}-\frac{\partial^2\Lambda}{\partial\beta^2} & -\frac{\partial^2\Lambda}{\partial\beta\partial\eta}\\ -\frac{\partial^2\Lambda}{\partial\eta\partial\beta} & -\frac{\partial^2\Lambda}{\partial\eta^2}\end{bmatrix}^{-1}_{\beta=\hat{\beta},\eta=\hat{\eta}}. \tag{2.7}$$

It is well known that the $\hat{\underline{\Theta}}=[\hat{\beta},\hat{\eta}]$ has an asymptotically multivariate normal distribution with mean $\underline{\Theta}=[\beta,\eta]$ and variance–covariance matrix $\Sigma_{\hat{\underline{\Theta}}}$.

Let $G(\underline{\Theta})$ be a function of $\underline{\Theta}$. Then, the variance of $G(\hat{\underline{\Theta}})$ can be approximated by:

$$Var(G(\hat{\underline{\Theta}}))=\left[\frac{\partial G(\underline{\Theta})}{\partial\beta}\right]^2Var(\hat{\beta})+\left[\frac{\partial G(\underline{\Theta})}{\partial\eta}\right]^2Var(\hat{\eta})+2\frac{\partial G(\underline{\Theta})}{\partial\beta}\frac{\partial G(\underline{\Theta})}{\partial\eta}Cov(\hat{\beta},\hat{\eta}). \tag{2.8}$$

Using the large sample normal approximation, the approximate one-sided lower or upper confidence bound on $G(\underline{\Theta})$ at the confidence level of $1-\alpha$ is (Meeker & Escobar, 1998):

$$G(\underline{\Theta})_{U,L}=G(\hat{\underline{\Theta}})\pm K_{\alpha}\sqrt{Var\left(G\left(\hat{\underline{\Theta}}\right)\right)}, \tag{2.9}$$