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# Solving the Pareto front for multiobjective Markov chains using the minimum Euclidean distance gradient-based optimization method

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## Highlights

- Present a novel method based on minimizing the Euclidean distance.
- Introduce Tikhonov's regularization method for ensuring strict-convexity of Pareto front.
- Propose a linear constraints over the nonlinear problem employing the $c$-variable method.
- Generate an even representation of the entire Pareto surface employing a distance restriction.
- Present an algorithm for solving multi-objective Markov chains problems.

## Abstract

A novel method based on minimizing the Euclidean distance is proposed for generating a well-distributed Pareto set in multi-objective optimization for a class of ergodic controllable Markov chains. The proposed approach is based on the concept of strong Pareto policy. We consider the case where the search space is a non-strictly convex set. For solving the problem we introduce the Tikhonov's regularization method and implement the Lagrange principle. We formulate the original problem introducing linear constraints over the nonlinear problem employing the $c$-variable method and constraining the cost-functions allowing points in the Pareto front to have a small distance from one another. As a result, the proposed method generates an even representation of the entire Pareto surface. Then, we propose an algorithm to compute the Pareto front and provide all the details needed to implement the method in an efficient and numerically stable way. As well, we prove the main Theorems for describing the dependence of the saddle point for the regularizing parameter and analyzes its asymptotic behavior. Moreover, we analyze the step size parameter of the Lagrange principle and also its asymptotic behavior. The suggested approach is validated theoretically and verified by a numerical example related to security patrolling that present a technique for visualizing the Pareto front.


*Keywords:* Multi-objective optimization; Markov chains; Pareto front; Strong Pareto policies; Euclidean distance

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## 1. Introduction

### 1.1. Brief review

In the traditional optimal Markov control problem the main goal is to find an optimal policy (strategy) that optimizes a single objective function [24]. The optimal policy is a point where the given objective function assumes its minimum, if a solution exists. On the other hand, the multi-objective optimization problem (MOP) is related to optimize several functions at the same time. The optimum policy of an individual function is different from the optima policies of the other objective functions.

The fundamental problem is to construct the Pareto front composed of an infinite number of the so-called non-dominated points. In particular, a policy that minimizes the objective function in the sense of Pareto is said to be a Pareto policy. An *utopia point* is determined by the infimum of the objective function. A key issue for constructing the Pareto set is to find the strong Pareto policies, determined by objective function, that are closest to the utopia policies in the sense of the usual Euclidean norm.

Multi-objective optimization is a very interesting area of research. For a survey of different types nonlinear MOPs we refer to [17] and [16] and, in the linear case to [13] and [29]. A different approach to tackle the problem, advantageous in the situation where the MOP is discrete, is by using Evolutionary Algorithms (see [34,6,5,9,14]) or Particle Swarm Optimization (see [8,18,15,12]. A method which is based on a stochastic approach is presented in [28], continuation or homotopy in [25,26], and a geometrically motivated methods are in [4,27]. Another way to compute the entire Pareto set is to use subdivision techniques (see [7]). Different methods focus on defining algorithms producing a well-distributed Pareto set [21,22,33].

The algorithms for finding the Pareto set presented in the literature are structurally stable, in the sense that they are able to build talented representations of the Pareto front when the functions are convex (strictly convex). Most of the existing solutions, supported by a local search approach based on classical linear and nonlinear programming, suppose that always a solution exists. However, in general there is a serious problem: the search space is in most of the cases a non-strictly convex set. This kind of behavior represents a typical situation (it is not artificially achieved for unrealistic objective functions).

Tikhonov regularization [31,30] is one of the most popular approaches to solve discrete ill-posed of the minimization problem

$$\min_x \|Ax - b\| . \tag{1}$$

The method seeks to determine a useful approximation of $x$ by replacing the minimization problem (1) by a penalized least-squares problem of the form

$$\min_x \|Ax - b\|^2 + \delta \|Lx\|^2$$

with regularization parameter $\delta > 0$ is chosen to control the size of the solution vector.

### 1.2. Contributions of this paper

We consider the problem of minimizing the Euclidean distance to a given affine space

$$\min \tfrac{1}{2} \|x\|^2 : Ax = b.$$

The main problem is proving the existence and characterization of strong Pareto policies. The existence of Pareto, weak Pareto, and proper Pareto policies is easy to compute because it can be obtained using the traditional scalarization method. We present an original formulation in terms of a coupled nonlinear programming problem implementing the Lagrange principle. For solving the existence and characterization of strong Pareto policies we employed the Tikhonov's regularization method. Regularization refers to a process of introducing additional information in order to solve an ill-posed problem. Specifically, Tikhonov regularization is a trade-off between fitting the data and reducing a norm of the solution ensuring the convergence of the objective functions to a local Pareto optimal policy. Each equation in this system is an optimization problem for which the necessary condition of a minimum is solve using the projection gradient method. For continuation proposes we restrict the cost-functions allowing points in the Pareto front to have a