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New improved convergence analysis for the secant method

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Abstract

We present a new convergence analysis, for the secant method in order to approximate a locally unique solution of a nonlinear equation in a Banach space. Our idea uses Lipschitz and center–Lipschitz instead of just Lipschitz conditions in the convergence analysis. The new convergence analysis leads to more precise error bounds and to a better information on the location of the solution than the corresponding ones in earlier studies. Numerical examples validating the theoretical results are also provided in this study.

Keywords: Secant method; Banach space; Majorizing sequence; Divided difference; Fréchet-derivative

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A vast number of problems from Applied Science including engineering can be brought in a form like (1.1) using mathematical modeling [5,12,14,17]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations cannot be found in closed form. This is the main reason why the most commonly used solution methods are iterative. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence analysis one derives convergence criteria from the information around an initial point whereas in the local analysis one finds estimates of the radii of convergence balls from the information around a solution.

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We consider the secant method defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - A_n^{-1} F(x_n) \quad (x_{-1}, x_0 \in \mathcal{D}), \qquad A_n = \delta F(x_n, x_{n-1}), \tag{1.2}$$

where mapping $\delta F(x, y) : \mathcal{D} \times \mathcal{D} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a consistent approximation to $F'(x)$ and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ $(x, y \in \mathcal{D})$, is the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ [17,20].

A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypotheses. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$. These are our objectives in this paper.

The secant method, also known under the name of Regula Falsi or the method of chords, is one of the most used iterative procedures for solving nonlinear equations. According to A.N. Ostrowski [21], this method is known from the time of early Italian algebraists. In the case of equations defined on the real line, the secant method is better than Newton's method from the point of view of the efficiency index [5]. The secant method was extended for the solution of nonlinear equations in Banach Spaces by A.S. Sergeev [26] and J.W. Schmidt [25].

The simplified secant method defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - A_0^{-1} F(x_n), \quad (x_{-1}, x_0 \in D)$$

was first studied by S. Ulm [27]. The first semilocal convergence analysis was given by P. Laasonen [22]. His results were improved by F.A. Potra and V. Pták [24–26]. A semilocal convergence analysis for general secant-type methods was given in general by J.E. Dennis [12]. Bosarge and Falb [10], Dennis [12], Potra [24–26], Argyros [6,7,5,8,9], Hernández et al. [15] and others [11,16,20,29,30], have provided sufficient convergence conditions for the secant method based on Lipschitz-type conditions on δF.

The use of Lipschitz and center-Lipschitz conditions is one way used to enlarge the convergence domain of different methods. This technique consists on using both conditions together instead of using only the Lipschitz one which allow us to find a finer majorizing sequence, that is, a larger convergence domain. It has been used in order to find weaker convergence criteria for Newton's method by Argyros in [8]. Gutiérrez et al. in [14] give sufficient conditions for Newton's method using both Lipschitz and center-Lipschitz conditions, Magreñán in [19] for the damped Newton's methods and Amat et al. in [2,4] or García-Olivo in [13] for other methods.

Here using Lipschitz and center-Lipschitz conditions, we provide a new semilocal convergence analysis for (1.2). It turns out that our error bounds and the information on the location of the solution are more precise (under the same convergence condition (see (2.1))) than the old ones given in earlier studies such as [1,16,18,20,23,22,24,25,30,28]. The rest of the paper is organized as follows: The semilocal and local convergence analysis of the secant method is presented in Section 2. Numerical examples are provided in Section 3.

2. Convergence analysis of the secant method

We need an auxiliary result on majorizing sequences for the secant method (1.2).

Lemma 2.1. *Let $L_{-1} \geq 0$, $L_0 \geq 0$, $L_1 \geq 0$, $L_2 \geq 0$, $N_{-1} > 0$, $N_0 \geq 0$, $c \geq 0$, $\eta \geq 0$ be given parameters with L_{-1} and L_0 not being zero at the same time. Let α be the smallest root of the polynomial*

$$p(s) = L_0 s^3 + (N_0 + L_{-1}) s^2 + (N_{-1} - N_0) s - N_{-1}$$

in the interval $(0, 1)$. *Define parameters*

$$s_{-1} = 0, \qquad s_0 = c, \qquad s_1 = c + \eta$$

$$s_2 = s_1 + \frac{L_2 (s_1 - s_0)^2 + L_1 (s_1 - s_0)(s_0 - s_{-1})}{1 - (L_2 (s_1 - s_0) + L_1 s_0)}$$

and

$$\alpha_1 = \frac{N_0 (s_2 - s_1) + N_{-1} (s_1 - s_0)}{1 - (L_0 (s_2 - s_0) + L_{-1} s_1)}.$$