Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 111 (2015) 1–16
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Absorbing boundary conditions and geometric integration: A case study for the wave equation

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Received 15 March 2013; received in revised form 23 July 2014; accepted 26 November 2014
Available online 11 December 2014

Abstract

This paper is concerned about the confluence of two subjects of the numerical solution of time evolution PDEs: numerical methods that preserve geometric properties of the flow and the use of absorbing boundary conditions to reduce the computation to a finite domain. This confluence is studied with special attention to the time stability of the resulting full discretization. For this, the stability regions of the time integrators are revisited. Since geometric methods are not always A-stable, it is necessary a suitable behavior of the real part of the eigenvalues of the spatially discretized problem to avoid in practice any time instability. A deep study is carried out for the case of the one dimensional wave equation when it is discretized with finite differences, showing that this suitable behavior happens. Numerical experiments confirming the previous results are included.

Keywords: Wave equation; Absorbing boundary conditions; Geometric time integration; Time stability

1. Introduction

The use of geometric time integrators [14,20] not only leads to better qualitative properties of the numerical solution, but also to a better accuracy when a long time computation is made. There are a lot of numerical integrators with some of these properties: symplectic integrators for Hamiltonian systems, symmetric integrators for reversible systems and methods designed to preserve first integrals.

It is quite common that the problem is defined in an unbounded domain when it is desirable to preserve its geometric properties. In this case, it is also necessary to make the numerical computation in a finite domain with suitable artificial boundary conditions [4,5,10–13,22]. There are many possibilities to define these conditions at the boundary, but we are interested in the so called absorbing boundary conditions (ABCs) which are designed to achieve small reflections inside the computational domain along with other good properties as the local character and easy implementation.

Apparently, a paradox emerges when geometric integration and ABCs are simultaneously considered. On the one hand, the original problem must be conservative in order to use geometric integrators and, on the other hand, the

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http://dx.doi.org/10.1016/j.matcom.2014.11.021
0378-4754/

incorporation of ABCs converts the original problem into a strongly dissipative system because of the absorption of every part of the solution which arrives at the boundary. Then, it could be thought that it is absurd to use geometric methods in this case. However, the geometric properties of the solution are still relevant inside the computational domain.

In this situation, the stability of the time integration must be addressed. When the ABCs are incorporated, the eigenvalues of the discretized differential operator are complex numbers with, in general, non vanishing real part and with an increasing size when the spatial discretization is refined. Therefore, we can deduce that the stability analysis which has already been carried out in the literature for the conservative case, where the eigenvalues are purely imaginary complex numbers (see for example [17]), is not suitable for the case which we are interested in.

We address this question by considering as case study the one dimensional linear wave equation

$$u_{tt} = u_{xx}, \quad x \in \mathbb{R},\ t \geq 0. \tag{1}$$

The application of symplectic methods to the wave equation with absorbing boundary conditions, and the subsequent near preservation energy is not new, but it was published in [7,18]. In these works the authors studied the Eq. (1) in the interval $[0, L]$ with the transparent boundary conditions $u_t(0,t) = u_x(0,t)$, $u_t(L,t) = -u_x(L,t)$. This problem is discretized in time and space using the Preissman box scheme.

Our approach is different, since we start considering a spatial discretization of (1). Let $h > 0$ be a given spatial step, for real and fixed a, we take the nodes $x_j = a + jh$ for $j \in \mathbb{Z}$. Using second order finite differences to approximate the second derivative in space, we have

$$\frac{d^2U_j}{dt^2} = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}, \quad j \in \mathbb{Z},\ t \geq 0, \tag{2}$$

where $U_j \approx u(x_j)$.

We denote the computational window by $[a, b] = [x_0, x_N] = [a, a + Nh]$, where $h = (b - a)/N$ and $N \in \mathbb{N}$. Nonlocal transparent boundary conditions (TBCs), associated to the space discretized wave equation given by (2), have been obtained in Proposition 3.2 of [16]. In Fourier variables, and for small enough ωh, they are given by

$$\left.\begin{array}{rcl} \widehat{u}_0(\omega) & = & r_1(\omega h)\,\widehat{u}_1(\omega), \\ \widehat{u}_N(\omega) & = & r_1(\omega h)\,\widehat{u}_{N-1}(\omega), \end{array}\right\} \tag{3}$$

at x_0 and x_N respectively, where

$$r_1(z) = 1 - \frac{z^2}{2} - iz\left(1 - \left(\frac{z}{2}\right)^2\right)^{1/2}. \tag{4}$$

The ABCs are deduced by approximating the function $r_1(\omega h)$ by using Taylor or Padé expansions and taking the inverse Fourier transform in order to deduce the ABCs in the original variables. We will use the notation ABC(p, q) for the ABCs obtained when we use the Padé expansion given by a rational function $p_1(\omega h)/p_2(\omega h)$, where p_1 and p_2 are polynomial functions with degrees p and q respectively. In this case, we define the order of absorption as the number $p + q + 1$ (this definition corresponds to the one used in [4,5] and it is slightly different from the one used in [16], where the order of absorption is defined as $p + q$).

This paper is written considering ABC$(2, 2)$, which have fifth order of absorption. In this way, we obtain an ordinary differential system which may be written in the form

$$\frac{d^2u_h}{dt^2} = A_h u_h + B_h \frac{du_h}{dt}. \tag{5}$$

Here, A_h and B_h, given in Section 4, are matrices which dimension depends on the parameter h.

The following step is to rewrite (5) as a first order ordinary differential system

$$\frac{d}{dt}\begin{bmatrix} u_h \\ v_h \end{bmatrix} = \mathcal{A}_h \begin{bmatrix} u_h \\ v_h \end{bmatrix}, \tag{6}$$