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Sensitivity analysis and model order reduction for random linear dynamical systems

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Abstract

We consider linear dynamical systems defined by differential algebraic equations. The associated input–output behaviour is given by a transfer function in the frequency domain. Physical parameters of the dynamical system are replaced by random variables to quantify uncertainties. We analyse the sensitivity of the transfer function with respect to the random variables. Total sensitivity coefficients are computed by a nonintrusive and by an intrusive method based on the expansions in series of the polynomial chaos. In addition, a reduction of the state space is applied in the intrusive method. Due to the sensitivities, we perform a model order reduction within the random space by changing unessential random variables back to constants. The error of this reduction is analysed. We present numerical simulations of a test example modelling a linear electric network.

Keywords: Linear dynamical systems; Differential algebraic equations; Sensitivity analysis; Model order reduction; Polynomial chaos

1. Introduction

Mathematical modelling of technical applications often results in systems of differential algebraic equations (DAEs). Examples are models of electric circuits or multibody dynamics, see [22,36]. We consider linear dynamical systems, which represent DAEs or ordinary differential equations (ODEs). A Laplace transformation reveals the input–output behaviour in the frequency domain.

The physical parameters of the system may exhibit uncertainties. We substitute the parameters by independent random variables for modelling the uncertainties. The number of parameters is often large, since the technical application involves many components. Thus our aim is to obtain an uncertainty quantification in case of a high-dimensional random space. The large number of random variables makes a numerical simulation by standard methods too costly. Thus we require techniques of model order reduction (MOR) to decrease the complexity.

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Appropriate MOR methods to reduce the dimension of the state space for linear dynamical systems already exist based on the transfer function, see [2,6,18,34]. Our idea is to analyse the transfer function also for a reduction of a high-dimensional random space in the stochastic modelling. Concepts for a variance-based sensitivity analysis are available for general functions depending on random variables, see [37].

We compute the required total sensitivity coefficients approximately by the expansions of the generalised polynomial chaos (gPC), following [40]. For this purpose, we investigate nonintrusive approaches based on quadrature as well as intrusive approaches resulting from the stochastic Galerkin method, see [41,46]. Moreover, an MOR of the state space is considered for the huge systems in the intrusive method. Techniques based on gPC have been applied successfully to nonlinear ODEs in [3,4,26] and to linear or nonlinear DAEs in [25,27,28,32,33].

Sobol [37] suggests to replace random variables with relatively small total sensitivities by constants. We apply this approach to reduce the dimension of the random space observing the sensitivity of the transfer function. While only sparse grids or sampling methods are able to tackle high-dimensional problems, the reduced order model can be resolved by highly accurate tensor-product grids now. We analyse the error of the reduction in the probability space and its interaction between the frequency domain and the time domain. Numerical simulations of a test example confirm the efficiency of this MOR in the random space. Alternatively, methods based on compressed sensing, see [15], or least angle regression, see [9], produce a sparse representation within the gPC, which can be seen as another form of MOR.

The paper is organised as follows. The type of linear problems and the stochastic modelling is introduced in Section 2. The gPC expansions and the related numerical methods are specified in Section 3. We outline the sensitivity analysis and define our MOR approach in Section 4. The error of this MOR is estimated stochastically. In Section 5, we present results of numerical simulations using the derived strategy. An Appendix contains the proofs of the theorems stated in Section 4.

2. Dynamical systems with random parameters

In this section, we define the problems to be investigated. Therein, the dependence of linear dynamical systems on physical parameters is substantial.

2.1. Linear dynamical systems

We discuss linear systems of the form

$$\begin{aligned} C(p)x'(t,p) + G(p)x(t,p) &= Bu(t) \\ y(t,p) &= Lx(t,p) \end{aligned} \tag{1}$$

for $t \in I_t$ with $I_t = [0, t_{\text{end}}]$ or $I_t = [0, \infty)$. The matrices $C(p), G(p) \in \mathbb{R}^{N \times N}$ depend on parameters $p \in \Pi \subseteq \mathbb{R}^Q$. Thus the state variables $x : I_t \times \Pi \to \mathbb{R}^N$ are also parameter-dependent. Input signals $u : I_t \to \mathbb{R}^{N_{\text{in}}}$ are introduced via a constant matrix $B \in \mathbb{R}^{N \times N_{\text{in}}}$. We define output signals $y : I_t \times \Pi \to \mathbb{R}^{N_{\text{out}}}$ by the state variables using a constant matrix $L \in \mathbb{R}^{N_{\text{out}} \times N}$.

Often the matrix $C(p)$ is singular for all $p \in \Pi$, i.e., the system (1) represents DAEs. We assume that the associated matrix pencil $C(p) + \lambda G(p)$ is regular for each $p \in \Pi$ to guarantee existence and uniqueness of solutions for initial value problems. Moreover, let the matrix pencil be regular in the limit case $\lambda \to \infty$ to ensure the applicability of time integration techniques, cf. [23]. If $G(p)$ is regular for all $p \in \Pi$, then we obtain existence and uniqueness of stationary solutions for each p.

Without loss of generality, we assume initial values $x(0, p) = 0$, since the transformation $z(t, p) := x(t, p) - x(0, p)$ is applied otherwise. In addition, we consider the case $u(0) = 0$ for the input. Let $X(s, p)$, $Y(s, p)$, $U(s)$ for $s \in \Sigma \subseteq \mathbb{C}$ be the Laplace transforms of the state variables, the output signals and the input signals, respectively. Often just the imaginary axis $s = \mathrm{i}\omega$ for frequencies $\omega \in \mathbb{R}$ is considered. A transformation of the linear dynamical system (1) into the frequency domain yields the input–output relation

$$Y(s,p) = H(s,p)U(s) \tag{2}$$

with the transfer function $H : \Sigma \times \Pi \to \mathbb{C}^{N_{\text{out}} \times N_{\text{in}}}$ defined by

$$H(s,p) := L\left(G(p) + sC(p)\right)^{-1} B \tag{3}$$