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Original Article

Invariant manifolds for nonsmooth systems with sliding mode

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Abstract

Invariant manifolds play an important role in the study of Dynamical Systems, since they help to reduce the essential dynamics to lower dimensional objects. In that way, a bifurcation analysis can easily be performed. In the classical approach, the reduction to invariant manifolds requires smoothness of the system which is typically not given for nonsmooth systems. For that reason, techniques have been developed to extend such a reduction procedure to nonsmooth systems. In the present paper, we present such an approach for systems involving sliding motion. In addition, an analysis of the reduced equation shows that the generation of periodic orbits through nonlinear perturbations which is usually related to Hopf bifurcation follows a different type of bifurcation if nonsmooth elements are present, since generically symmetry is broken by the nonsmooth terms.

Keywords: Invariant manifold; Sliding motion; Nonlinear piecewise dynamical systems; Non-smooth systems; Invariant cones; Periodic orbits; Generalized Hopf bifurcation

1. Introduction

The use of invariant manifolds, in particular of center manifolds, has been established as a key tool for analyzing the dynamics of high dimensional dynamical systems, since it allows a reduction to systems of low dimension (see [5]). Following that approach, a bifurcation analysis can be carried out with the aim, for example, to study the generation of periodic orbits.

For linear systems, the reduction corresponds to a decomposition into invariant subspaces characterized by the eigenvalues; the notion of invariant manifolds has been developed as the analog for nonlinear systems. To carry out the reduction, appropriate smoothness of the system is required, hence, that approach fails for nonsmooth systems. Nevertheless, it has been possible to define invariant sets for nonsmooth systems as well (see [11,12,16]). For planar systems there is no need for a reduction; Hence, planar piecewise smooth systems have been studied first for example in [8,15,17] to understand the modifications due to nonsmoothness. For piecewise linear systems, they appear as the surface of invariant cones, hence, two-dimensional objects which can be used to describe the dynamics of the full

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systems. In relevant cases, the cones are attractive, but the dynamics on the cone can be asymptotically stable, stable or unstable. The existence of such invariant cones for piecewise linear systems consisting of periodic orbits has already been observed in [1–4]. For smooth systems, these cones become flat, hence, degenerate to planes. The flow within the plane, for example, may change from a stable focus through a center to an unstable focus if an appropriate parameter is varied. In the case of linear systems, this mechanism provides the basis for Hopf bifurcation; if the system is perturbed by higher order terms periodic orbits might occur (see [9,14]).

In [16], this approach has been generalized to piecewise smooth systems. Starting with a piecewise linear system as basic system, it has been shown that the corresponding invariant cones will be deformed to a cone-like surface if higher order terms are added. Using standard techniques such as the Hadamard-Graph-Transform, the existence of such an invariant "manifold" has been established, but in the first approach only for systems without sliding motion. Since sliding motion is a key feature of nonsmooth systems, that setting has to be taken into account. Although, the dimension of the flow is reduced by the sliding part another difficulty arises as the flow within the sliding area is nonlinear even for piecewise linear systems. To set up an appropriate basic system, it helps that its flow is always homogeneous. In addition, situations can be characterized, where the sliding flow is linear. For such special cases the extension of the manifold notion is straight forward.

In the present paper, we will show that the existence of invariant lower dimensional manifolds capturing the essential dynamics can also be obtained, for systems involving sliding. Using the reduced system, bifurcation analysis can be performed by analyzing a one-dimensional Poincaré-map defined on an invariant curve, where a non-trivial fixed point corresponds to an periodic orbit of the original system. At the first glance, this appears as a direct generalization of Hopf bifurcation. A detailed analysis of the Poincaré-map though elucidates some new features. Due to inherent symmetries of smooth systems in the case of Hopf bifurcation, it is possible to eliminate all nonlinear terms of even order so that generically third order terms turn out to be dominating for bifurcation (see for example [9,14]). For nonsmooth systems that kind of symmetry is broken, implying that already quadratic terms determine the bifurcation behavior.

More precisely, in [16], we proved that under certain attractivity and transversality conditions (see Section 3) piecewise nonlinear systems (PWNS) of the form

$$\dot{\xi} = \begin{cases} A^{+}\xi + g_{+}(\xi), & n^{\mathrm{T}}\xi > 0, \\ A^{-}\xi + g_{-}(\xi), & n^{\mathrm{T}}\xi < 0 \end{cases} \tag{1}$$

with constant matrices $A^{\pm}$ and nonlinear C^k-parts $g_{\pm}(\xi) = o(\| \xi \|)$, $k \geq 1$, exhibit an invariant cone-like surface if the corresponding piecewise linear system (PWLS)

$$\dot{\xi} = \begin{cases} A^{+}\xi, & n^{\mathrm{T}}\xi > 0, \\ A^{-}\xi, & n^{\mathrm{T}}\xi < 0 \end{cases} \tag{2}$$

possesses an invariant cone. The analysis in [16] was done in case of direct crossing, i.e., the dynamics on the cone does not involve sliding motion.

We proceed as follows: In Section 2, we characterize and study the system governing the dynamics in case of sliding mode, for piecewise linear (PWLS) and piecewise nonlinear systems (PWNS). Section 3 is dedicated to review shortly the situation without sliding mode, which is already treated in [16]. Supplementary to the conclusions in [16], we discuss, in Section 4, similar results in presence of sliding motion, which are proven in Section 6. In Section 5, we analyze the generation of periodic solution using the derived results. We compare the mechanism with the well-understood Hopf bifurcation in smooth systems and point out the crucial differences.

2. Sliding motion

2.1. Sliding mode in PWLS

We consider piecewise linear systems (2) in $\mathbb{R}^N$ consisting of 2 components separated by the hyperplane $\mathcal{M} = \{\xi \in \mathbb{R}^N \mid n^{\mathrm{T}}\xi = 0\}$ with constant matrices $A^{\pm}$ and normal vector n^{T}. Using $\rho(\xi) = n^{\mathrm{T}}A^{+}\xi \cdot n^{\mathrm{T}}A^{-}\xi$, we define the sliding