Available online at www.sciencedirect.com

ScienceDirect

Mathematics and Computers in Simulation 110 (2015) 144–154

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original Article

Convergence of a numerical method for the solution of non-standard integro-differential boundary value problems

M. Basile [a], E. Messina [a,*], W. Themistoclakis [b], A. Vecchio [b]

[a] Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli "Federico II", Via Cintia, I-80126 Napoli, Italy

[b] C.N.R. National Research Council of Italy, Istituto per Applicazioni del Calcolo "Mauro Picone", Sede di Napoli, via P. Castellino, 111, 80131 Napoli, Italy

Received 10 December 2012; received in revised form 10 September 2013; accepted 26 November 2013
Available online 7 December 2013

Abstract

In a recent paper we proposed a numerical method to solve a *non-standard* non-linear second order integro-differential boundary value problem. Here, we answer two questions remained open: we state the order of convergence of this method and provide some sufficient conditions for the uniqueness of the solution both of the discrete and the continuous problem. Finally, we compare the performances of the method for different choices of the iteration procedure to solve the non-standard nonlinearity.

MSC: 45J05; 34B40; 65L20

Keywords: Numerical solution of boundary value problems; Non-linear non-standard integro-differential equations; Half-line; Order of convergence; Uniqueness

1. Introduction

In this paper we consider the non-standard second order integro-differential equation

$$\begin{cases} v(y)g(y) - \int_0^{+\infty} k(x)g(x)dx[D(y)g'(y)]' = p(y) \\ g'(0) = 0, \quad \lim_{y\to+\infty} g(y) = 0 \end{cases}, \quad y\geq 0, \tag{1}$$

which is a simplified model for a problem of kinetic theory of dusty plasmas (see [4,5,8,9]). Eq. (1) has been investigated both from a theoretical [1] and a numerical [2] point of view by the same authors, who proposed and analyzed a numerical method tuned to the *non-standard* nature of the problem itself. The integro-differential problem (1) is non-standard in the sense that the coefficients of the differential terms of the unknown function depend on the unknown itself, by means of an integral over the semi-axis. For this reason, the method proposed in [2] consists of two steps: the discretization of the differential and integral terms by means of finite difference formulas and quadrature formulas respectively, the resolution of the resulting non-linear system by means of the bisection iterative process. In the same paper the analysis

* Corresponding author. Tel.: +39 081675652.
E-mail addresses: mariateresa.basile@unina.it (M. Basile), eleonora.messina@unina.it (E. Messina), woula.themistoclakis@cnr.it (W. Themistoclakis), antonia.vecchio@cnr.it (A. Vecchio).

0378-4754/$36.00
http://dx.doi.org/10.1016/j.matcom.2013.11.003

of convergence has been provided, but, while the numerical experiments clearly show a convergence order two, this remained an open problem from a theoretical point of view. In this paper we address this problem by proving that, under suitable assumptions, the order of convergence of the method in [2] is in fact two. Furthermore, in [1] the existence of a solution of (1) was stated, but its uniqueness was only a conjecture confirmed by the numerical experiments carried out in [2]. Here, we furnish sufficient conditions for the continuous and the discrete problem to have a unique solution and we show how, in this case, the order convergence extends to numerical methods based on any iterative procedure applied for solving the nonlinearity of the discrete problem.

In order to describe the numerical method we write down problem (1) as

$$\begin{cases} \nu(y)g(y,q) - q[D(y)g'(y,q)]' = p(y) \\ g'(0,q) = 0, \quad \lim\limits_{y\to+\infty} g(y,q) = 0 \end{cases}, \quad y\geq 0,\ q>0, \tag{2}$$

$$q = f(q), \quad f(q) := \int_0^{+\infty} k(x)g(x,q)dx, \tag{3}$$

where the parametric problem (2) coincides with (1) when q is a fixed point of the nonlinear function $f(q)$ defined in (3).

Section 2 contains a synthesis of the investigations carried out in [1,2]; in Section 3, using an idea developed in [6,7], we analyze the order of convergence of the numerical method and in Section 4 we establish some sufficient assumptions for the uniqueness of the solution. In Section 5, which is devoted to the numerical experiments, we illustrate the convergence of the numerical method based on other iterative processes and in Section 6 some concluding remarks are reported.

2. Background

We denote by $BC^r[0,+\infty)$ the space of all continuous and bounded functions on $[0,+\infty)$ having continuous and bounded derivatives up to order r. Moreover, $C^r[0,+\infty)$ denotes the usual space of continuous and differentiable functions up to order r. From now on we assume that the following hypotheses on the functions involved in (1) hold

(h_1) $D\in BC^3[0,+\infty)$, $\nu\in C^2[0,+\infty)$, $k\in C^3[0,+\infty)$, $p\in BC^2[0,+\infty)$,
(h_2) $0<D_{inf}\leq D(y)\leq D_{sup}$, $|D'(y)|\leq D_1$, $y\geq 0$,
(h_3) $0<\nu_{inf}\leq \nu(y)\leq \nu_{sup}$, $|\nu^{(i)}(y)|\leq \nu_i$, $\quad i=1,2, \quad y\geq 0$,
(h_4) $0\leq p(y)\leq P$, $\quad y\geq 0$,
(h_5) $\lim_{y\to+\infty} p(y)=0$,
(h_6) $\int_0^{+\infty} p(y)dy < +\infty$,
(h_7) $k(y)\geq 0$, $\quad y\geq 0$,
(h_8) $\int_0^{+\infty} |k^{(i)}(x)|dx < +\infty$, $\quad i=0,\ldots,3$,

with $D_{inf}, D_{sup}, D_1, \nu_{inf}, \nu_{sup}, P$ positive constants and $k(y)$ and $p(y)$ not identically zero. These assumptions summarize the ones set in [1,2] where we prove the existence of a solution $g(y)$ of problem (1) and propose a numerical method to approximate it.

In order to solve numerically problems (2)–(3) we fix the step length $h>0$, consider problem (2) on $[0, T]$, with $T=Nh$ sufficiently large, and a uniform mesh on it:

$$\Pi_h : 0 = y_0 < y_1 < y_2 < \ldots < y_{N-1} < y_N = T, \quad y_i = ih, \quad i = 0,\ldots,N. \tag{4}$$

The meaning of T *sufficiently large* is deeply explained in [2] (Sections 3–5), where we also show how to compute it.

Since the solution $g(y)$ of (1) tends to zero as $y\to\infty$, we are allowed to add the following assumption

(h_9) T is large enough to have $|g(y,q)|=O(h^4)$, for all $y\geq T$.

Let us set

$$g_i = g_i(q) \approx g(y_i,q), \quad i=0,\ldots,N-1,\ g_N=0, \tag{5}$$