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Constructing adaptive generalized polynomial chaos method to measure the uncertainty in continuous models: A computational approach

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Abstract

Due to errors in measurements and inherent variability in the quantities of interest, models based on random differential equations give more realistic results than their deterministic counterpart. The generalized polynomial chaos (gPC) is a powerful technique used to approximate the solution of these equations when the random inputs follow standard probability distributions. But in many cases these random inputs do not have a standard probability distribution. In this paper, we present a step-by-step constructive methodology to implement directly a useful version of *adaptive* gPC for arbitrary distributions, extending the applicability of the gPC. The paper mainly focuses on the computational aspects, on the implementation of the method and on the creation of a useful software tool. This tool allows the user to easily change the types of distributions and the order of the expansions, and to study their effects on the convergence and on the results. Several examples illustrating the usefulness of the method are included.

Keywords: Random differential equations; Adaptive polynomial chaos; Computing

1. Introduction and motivation

Indealing with random differential equations, a considerable number of useful methods have been developed [11,20,3,2,15,14,18,21,1,6,4]. Here, we are specifically interested in the generalized Polynomial Chaos method (gPC) since it has been shown to be particularly effective for a number of problems [10,26,7]. gPC was proposed by Xiu and

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Karniadakis [26] as a remarkable generalization of the pioneering work by N. Wiener [25], which is usually referred to as the *homogeneous chaos* or Hermite-PC.

The application of gPC requires that every random model parameter belongs to some of the standard probability distributions. Otherwise, it has to be approximated by some of them. In practice, this approximation is usually made by using Gaussian random variables. Although in many practical cases the previous approach can be legitimized by the Central Limit Theorem, it is desirable to investigate alternative approaches since sometimes such approximations cannot be assumed or, even if it can be assumed, the improvement of the accuracy of the results constitutes a goal itself.

The aim of this paper is to present a method that provides researchers, who do not know the foundations of gPC, a step-by-step computational approach technique to implement an adaptive gPC method to be used in random continuous models (random differential equations). In this method the random inputs do not need to belong necessarily to some of the standard probability families. This point has great interest since, in practice, most of the random variables constructed to model real phenomena likely do not fit the standard distributions, and therefore the proposed method extends the applicability of the gPC. This approach requires the construction of probability density functions of input random variables that, according to gPC method, will play the same role as the weighting functions do [26]. Notice that methods to build such probability density functions from sampled data are well-developed [16,19,23]. Several illustrative examples including practical applications showing the usefulness of the proposed method are presented in this paper.

This paper is organized as follows. For the sake of clarity, in Section 2 we summarize the gPC method focusing on its application to solve random differential equations. Section 3 is devoted to introducing a variation of gPC *adapted* to the case where every random model parameter may have probability distributions different from the standard ones. A pseudo-code algorithm to compute the main statistical functions of the solution stochastic process to random differential equations is developed in Section 4, where significant computational comments are included. In Section 5, we apply adaptive gPC to solve several illustrative examples including both, test-problems and models appearing in applications. Finally, conclusions are drawn in Section 6.

2. gPC method: a short review focusing on the solution of random differential equations

For the sake of clarity in the subsequent developments, this section explains how gPC works in dealing with the solution of random differential equations

$$\mathfrak{D}(t, \boldsymbol{\xi}(\omega); y) = f(t, \boldsymbol{\xi}(\omega)), \tag{1}$$

where $\mathfrak{D}$ denotes a differential operator; $y = y(t, \boldsymbol{\xi}(\omega))$ is the solution stochastic process to be determined and $f(t, \boldsymbol{\xi}(\omega))$ is a forcing term. Notice that in the random differential equation (1) uncertainty is represented by $\boldsymbol{\xi}$ and it just enters through their coefficients and forcing term, although in practice it could also be considered via initial and/or boundary conditions.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and let us consider the set L^2 whose elements are second-order real random variables, that is, real random variables ζ having finite variance, or equivalently $\mathrm{E}\left[\zeta^2\right] < \infty$, where $\mathrm{E}[\cdot]$ denotes the expectation operator. The set L^2 endowed with the inner product $\langle \zeta_1, \zeta_2 \rangle = \mathrm{E}[\zeta_1 \zeta_2]$, is a Hilbert space, usually denoted by $\mathrm{L}^2(\Omega, \mathfrak{F}, P)$ [17]. The norm inferred by the above inner product determines the mean-square convergence.

gPC is a powerful technique to represent spectrally in the random dimension random variables ζ and stochastic processes $y(t)$ in $\mathrm{L}^2(\Omega, \mathfrak{F}, P)$. These representations are given by infinite random series defined in terms of certain orthogonal polynomial expansions $\{\Phi_i\}$ which depend on a number of random variables $\boldsymbol{\xi}(\omega) = (\xi_1(\omega), \xi_2(\omega), \ldots), \omega \in \Omega$,

$$\zeta = \sum_{i=0}^{\infty} \zeta_i \Phi_i(\boldsymbol{\xi}(\omega)), \qquad y(t) = \sum_{i=0}^{\infty} y_i(t) \Phi_i(\boldsymbol{\xi}(\omega)). \tag{2}$$

As we pointed out in the previous section, the choice of the trial basis $\{\Phi_i\}$ is crucial in dealing with random differential equations and it is what distinguishes homogeneous chaos from gPC. In the first one, $\{\Phi_i\}$ are just the Hermite polynomials defined in terms of Gaussian random variables ξ_i. Whereas in gPC, $\{\Phi_i\}$ belong to the