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Calculation of singular point quantities at infinity for a type of polynomial differential systems[☆]

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Abstract

The center problem at infinity is far to be solved in general. In this paper we develop a procedure to resolve it for a particular type of polynomial differential systems. The problem is solved by writing its concomitant differential equation in the complex coordinates introduced by Yirong Liu and by developing a new method of computation of the so called singular point quantities. This method is based on the transformation of infinity into the elementary origin. Finally, the investigation of center problem for the infinity of a particular family of planar polynomial vector fields of degree 5 is carried out to illustrate the main theoretical results. These involve extensive use of a Computer Algebra System, we have chosen to use *Mathematica*®.

Keywords: Infinity; Singular point quantity; Homeomorphic transformation; Recursive algorithm

1. Introduction and main results

One of the classical problems in the qualitative theory of planar analytic differential systems is the study of the local phase portrait at the singularities to characterize when a singular point is of center–focus type. The study of the singular points for planar analytic differential equations $\dot{x} = X(x)$ is almost totally solved. It is possible to know what is the behavior of the solutions of a planar analytic differential equation in a neighborhood of an isolated singular point, in all cases except in the *monodromic* case. Remember that this case is the one in which solutions of the differential equation turn around the singular point. Recall that a singular point is said to be of *center–focus type* if it is either a focus or a center. In what follows, this problem will be called the *center–focus problem* or the *monodromy problem*. The first problem appears when we have to decide whether a critical point is of monodromic type. This is usually done by the blow-up procedure. On the other hand, once we know that a singular point is of center–focus type, appears another classical problem usually called the *center problem* or the *stability problem*, which requires to

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distinguish between a focus and a center. A more difficult problem is to obtain efficient algorithms, there are some papers that do that, see [1,13,33,37].

For the case in which the differential matrix of the vector field at the singular point has imaginary eigenvalues, we know that the singular point is monodromic. If their real part are different from zero, then the singular point is a focus; while if their real part are zero, the singular point may be a center or a focus. Indeed this last case is strongly related with the celebrated *center–focus* problem. Recall that it consists in distinguish when a monodromic singular point (i.e., a singular point for which there are not orbits tending to it with a definite slope, in positive or negative time) is a focus (i.e., a singular point with a neighborhood where the orbits spiral toward or backward it) or a center (i.e., a singular point with a neighborhood where all the orbits are closed and periodic).

The center–focus problem for a non-degenerate singular point was theoretically solved by Lyapunov at the end of the XIX century, see [32]. In his work the author introduced the concept of the functions now known as *Lyapunov functions*, as well as the *Lyapunov constants* that give the stability of the point. The role of Lyapunov constants is the following: if for some $k > 1$ we have $V_3 = V_5 = \cdots = V_{2k+1} = 0$ and $V_{2k+1} \neq 0$, then the singular point is a focus, while if all the Lyapunov constants vanish, then the first return map is the identity map and the singular point is a center.

Vector fields with a non-degenerate singular point can be represented by equations of the following form

$$\begin{aligned} \frac{dx}{dt} &= -y + X(x, y), \\ \frac{dy}{dt} &= x + Y(x, y). \end{aligned} \tag{1.1}$$

In this case it can be proved that Lyapunov constants are polynomials in the coefficients of X and Y. One can therefore obtain necessary conditions for the origin to be a center by imposing the vanishing of the Lyapunov constants on these coefficients. This means that if X and Y are polynomials of a fixed degree (the most studied case), by Hilbert's Basis Theorem it is only necessary to compute a finite number of Lyapunov constants to obtain the characterization of the family. The calculation of Lyapunov constants is impossible by hand except in the simplest cases and computational methods have been developed. Research on the qualitatively theory of differential equations, and indeed on the whole subject of polynomial differential equations, has been stimulated considerably by the availability of Computer Algebra Systems. Though attempts have been made in the past to prove the necessity and sufficiency of such conditions in parallel, it is more fruitful and reliable to discuss the two separately. However, there are two significant obstructions to this procedure:

The computational obstruction: Although there have recently appeared several algorithms to compute Lyapunov constants (see for example [19,20,36] and reference therein), these algorithms all have to manipulate large and complicated algebraic expressions, especially for families of systems with a large number of parameters. Another case is that there are a lot of computational problems with systems with not so many parameters but with a lot of different monomials. Even using sophisticated tools for symbolic computational algebra, it is difficult to deal with the algebraic expressions which arise.

The sufficiency obstruction: Given a particular system of family of systems of type (1.1), one does not know how to determine if a large enough number of constants have been computed to guarantee that the system has a center at the origin. There is not known algorithmic approach to answer this question. Usually in this situation, one relies on a geometric or analytic property of the system which characterizes the presence of a center, such as a symmetry (time reversibility) or an integrable structure.

It is worthy of mention that in 1990, the authors of [27] unified the concepts focal values and saddle quantities into that of singular point quantities by transforming a real planar system into its concomitant complex system. And they described the equivalent relation of Lyapunov constants to singular point quantities. This transferred the problem of the calculation of focal values and Lyapunov constants to the calculation of singular point quantities. Afterwards, on the basis of [27], the authors of [11] deduced a practical linear recursion formula for computing singular point quantities.

For the situation in which the differential matrix of the vector field at the singular point has all the eigenvalues zero, but it is not identically zero (usually called the nilpotent case), the monodromic problem was solved by both of Andreev and Moussu, see [4,5,35]. Both the monodromy and the stability problem in the nilpotent case have been studied by Moussu [35]. His main result is to characterize the $\mathcal{C}^\infty$ normal form of the nilpotent centers. Berthier and Moussu in [7] studied the reversibility of the nilpotent centers.