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Original article

Factoring analytic multivariate polynomials and non-standard Cauchy–Riemann conditions

Tomas Recio [a], J.-Rafael Sendra [b], Luis-Felipe Tabera [a,*], Carlos Villarino [b]

[a] *Dpto. de Matemáticas, Universidad de Cantabria, 39071 Santander, Spain*

[b] *Dpto. de Física y Matemáticas, Universidad de Alcalá, 28871 Alcalá de Henares, Spain*

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Abstract

Motivated by previous work on the simplification of parametrizations of curves, in this paper we generalize the well-known notion of analytic polynomial (a bivariate polynomial $P(x, y)$, with complex coefficients, which arises by substituting $z \to x + \mathrm{i}y$ on a univariate polynomial $p(z) \in \mathbb{C}[z]$, i.e. $p(z) \to p(x + \mathrm{i}y) = P(x, y)$) to other finite field extensions, beyond the classical case of $\mathbb{R} \subset \mathbb{C}$. In this general setting we obtain different properties on the factorization, gcd's and resultants of analytic polynomials, which seem to be new even in the context of Complex Analysis. Moreover, we extend the well-known Cauchy–Riemann conditions (for harmonic conjugates) to this algebraic framework, proving that the new conditions also characterize the components of generalized analytic polynomials.

Keywords: Cauchy–Riemann conditions; Analytic polynomials; Hankel matrix; Factorization

1. Introduction

The well-known Cauchy–Riemann (in short: CR) equations provide necessary and sufficient conditions for a complex function $f(z)$ to be holomorphic (cf. [2,5]). One traditional framework to introduce the CR conditions is through the consideration of *harmonic conjugates*, $\{u(x, y), v(x, y)\}$, as the real and imaginary parts of a holomorphic function $f(z)$, after performing the substitution $z \to x + \mathrm{i}y$ (i denotes the imaginary unit), yielding $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}\, v(x, y)$. The Cauchy–Riemann conditions are a cornerstone in Complex Analysis and an essential ingredient of its many applications to Physics, Engineering, etc.

In this paper, we will consider two different, but related, issues. First, we will generalize CR conditions by replacing the real/complex framework by some more general field extension and, second, we will address – in this new setting – the specific factorization properties of conjugate harmonic polynomials. Let us briefly describe our approach to both topics in what follows.

* Corresponding author. Tel.: +34 942201431; fax: +34 942201402.
E-mail addresses: tomas.recio@unican.es (T. Recio), rafael.sendra@uah.es (J.-R. Sendra), taberalf@unican.es (L.-F. Tabera), carlos.villarino@uah.es (C. Villarino).
URLs: http://www.recio.tk (T. Recio), http://www2.uah.es/rsendra/ (J.-R. Sendra), http://personales.unican.es/taberalf/ (L.-F. Tabera).

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An *analytic polynomial* (a terminology taken from popular textbooks in Complex Analysis, see e.g. [2]), is a bivariate polynomial $P(x, y)$, with complex coefficients, which arises by substituting $z \to x + \mathtt{i}y$ on a univariate polynomial $p(z) \in \mathbb{C}[z]$, i.e. $p(z) \to p(x + \mathtt{i}y) = P(x, y)$. As stated above, a goal of our paper deals with generalizing CR conditions when suitably replacing the pair real/complex numbers by some other field extension. Remark that, in the formulation of the CR conditions, since we restrict ourselves to polynomials and rational functions, we can use the well-known concept of formal partial derivative of a polynomial or rational function, as in [9], chapter II, §17, without requiring the introduction of any topological concept or the idea of limit of a function at a point. For a simple example, take as base field $\mathbb{K} = \mathbb{Q}$ and then $\mathbb{K}(\alpha)$, with α such that $\alpha^3 + 2 = 0$. Then we will consider polynomials (or more complicated functions) $f(z) \in \mathbb{K}(\alpha)[z]$ and perform the substitution $z = x_0 + x_1\alpha + x_2\alpha^2$, yielding $f(x_0 + x_1\alpha + x_2\alpha^2) = u_0(x_0, x_1, x_2) + u_1(x_0, x_1, x_2)\alpha + u_2(x_0, x_1, x_2)\alpha^2$, where $u_i \in \mathbb{K}[x_0, x_1, x_2]$. Finally, we will like to find the necessary and sufficient conditions on a collection of polynomials $\{u_i(x_0, x_1, x_2)\}_{i=0,1,2}$ to be, as above, the *components* of the expansion of a polynomial $f(z)$ in the given field extension.

More generally, suppose $\mathbb{K}$ is a characteristic zero field, $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$, and α is an algebraic element over $\mathbb{K}$ of degree $r+1$. In this context we proceed, first, generalizing the concept of analytic polynomial as follows (see also [1,7], as well as Definition 1 below, for a more general, multivariate, definition):

A polynomial $p(x_0, \ldots, x_r) \in \mathbb{K}(\alpha)[x_0, \ldots, x_r]$ is called α-analytic if there exists a polynomial $f(z) \in \overline{\mathbb{K}}[z]$ such that

$$f(x_0 + x_1\alpha + \cdots + x_r\alpha^r) = p(x_0, \ldots, x_r).$$

We say that f is the generating polynomial of p. An analytic polynomial can be uniquely written as

$$p(x_0, \ldots, x_r) = u_0(x_0, \ldots, x_r) + u_1(x_0, \ldots, x_r)\alpha + \cdots + u_r(x_0, \ldots, x_r)\alpha^r,$$

where $u_i \in \mathbb{K}[x_0, \ldots, x_r]$. The polynomials u_i are called the $\mathbb{K}$-components of $p(x_0, \ldots, x_r)$.

The main result in this setting is the following statement (and its generalization to an even broader setting) expressing non-standard CR conditions (see Definition 3 and Theorem 15):

Let $\{u_0, \ldots, u_r\}$ be the $\mathbb{K}$-components of an α-analytic polynomial $p(x_0, \ldots, x_r)$. It holds that

$$\begin{pmatrix} \frac{\partial u_i}{\partial x_0} \\ \vdots \\ \frac{\partial u_i}{\partial x_r} \end{pmatrix} = H_i \cdot \begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \vdots \\ \frac{\partial u_r}{\partial x_0} \end{pmatrix}, \quad i = 0, \ldots, r$$

where H_i are the Hankel matrix introduced in Section 3. *And, conversely, if these equations hold among a collection of polynomials u_i, then they are the $\mathbb{K}$-components of an analytic polynomial.*

As expected, the above statement gives, in the complex case, the well-known CR conditions. In fact, let $\mathbb{K} = \mathbb{R}$, $\alpha = \mathtt{i}$, and $P(x_0, x_1) \in \mathbb{C}[x_0, x_1]$ be an analytic polynomial. If u_0, u_1 are the real and imaginary parts of P, the above Theorem states that

$$\begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \frac{\partial u_0}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \frac{\partial u_1}{\partial x_0} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u_1}{\partial x_0} \\ \frac{\partial u_1}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \frac{\partial u_1}{\partial x_0} \end{pmatrix}$$

which is a matrix form expression of the classic CR equations:

$$\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \quad \frac{\partial u_1}{\partial x_1} = \frac{\partial u_0}{\partial x_0}$$

It might be interesting to remark that the square matrix, expressing the above non-standard CR conditions, is a Hankel matrix (see [6] or chapter 7 in [8]), an ubiquitous companion of Computer Algebra practitioners.

A computational relevant context (and in fact our original motivation) of our work about generalized analytic polynomials is the following situation. Consider a rational function $f(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$ in several complex variables and with complex coefficients, then perform the substitution $\mathbf{z} = \mathbf{x} + \mathtt{i}\,\mathbf{y}$ and compute the real and imaginary parts of the resulting