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Original article

Finite element approximation of a surface–subsurface coupled problem arising in forest dynamics☆

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Abstract

We propose a finite element approximation for an evolution model describing the spatial population distribution of two salt tolerant plant species, such as mangroves, which are affected by inter- and intra-specific competition (Lotka–Volterra), population pressure (cross-diffusion) and environmental heterogeneity (environmental potential). The environmental potential and the Lotka–Volterra terms are assumed to depend on the salt concentration in the roots region, which may change as a result of mangroves ability for uptaking fresh water and leave the salt of the solution behind, in the saturated porous medium. Consequently, partial differential equations modeling the population dynamics on the surface are coupled with Darcy-transport equations modeling the salt and pressure–velocity distribution in the subsurface. We provide a numerical discretization based on a stabilized mixed finite element method for the transport-Darcy flow problem coupled to a finite element method for a regularized version of the cross-diffusion population model, which we use to numerically demonstrate the behavior of the system.

Keywords: Population dynamics; Cross-diffusion; Darcy flow; Stabilized mixed formulation; Finite element method

1. Introduction

We present a model for analyzing the spatial distribution evolution of two plant populations which are affected by

- competition for similar resources,
- population pressure, and
- environmental quality.

These conditionings are realized mathematically in the form of a time evolution drift-cross diffusion system of partial differential equations for the biomass densities, $u_i(\bar{x}, t) \geq 0$, of species $i = 1$ and $i = 2$, introduced by Shigesada et al. [22]:

$$\partial_t u_i - \operatorname{div} J_i = F_i(\cdot, u_1, u_2), \quad J_i = \nabla(c_i u_i + a_{ii} u_i^2 + a_{ij} u_i u_j) + d_i u_i \nabla \Phi, \tag{1}$$

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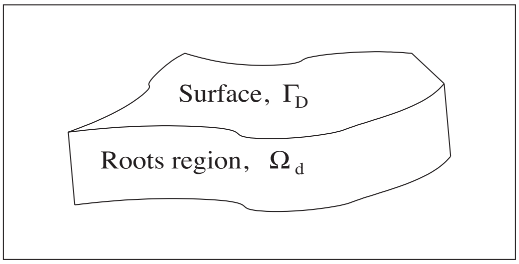

Fig. 1. Global domain for Problem P. On the surface, population dynamics takes place. In the subsurface, concentration-Darcy equations govern the unstable salt-water configuration. The link between both regions is, mainly, through the environmental potential, Φ.

where $i \neq j$, holding in $S_T = \Gamma_D \times (0, T)$, with $\bar{x} \in \Gamma_D \subset \mathbb{R}^{N-1}$, $N = 2$ or 3, open, bounded and with Lipschitz continuous boundary, $\partial\Gamma_D$, and for $t \in (0, T)$ the time, for an arbitrarily fixed $T > 0$. The spatial domain Γ_D represents the soil surface and is given as the top boundary of an N-dimensional bounded set, Ω, the subsurface domain.

The diffusion coefficients c_i and a_{ij} are non-negative constants, and $d_i \in \mathbb{R}$ ($i, j = 1, 2$). The source terms are of competitive Lotka–Volterra type

$$F_i(\bar{x}, t, u_1, u_2) = (\alpha_i(\bar{x}, t) - \beta_{i1}(\bar{x}, t)u_1 - \beta_{i2}(\bar{x}, t)u_2)u_i, \quad i = 1, 2, \tag{2}$$

where $\alpha_i \geq 0$ is the intrinsic growth rate of the i-specie, $\beta_{ii} \geq 0$ are the coefficients of intra-specific competition, β_{12}, $\beta_{21} \geq 0$ are those of inter-specific competition. Function $\Phi = \Phi(\bar{x}, t)$ is the environmental potential, modeling areas where the environmental conditions are more or less favorable [22,20]. We shall describe later how the Lotka–Volterra terms and the environmental potential are related to the evolving environment. The above system of equations is completed with non-flux boundary conditions and initial data:

$$J_i \cdot \nu = 0 \quad \text{on} \quad \partial\Gamma_D \times (0, T), \tag{3}$$

$$u_i(\cdot, 0) = u_i^0 \geq 0 \quad \text{on} \quad \Gamma_D, \tag{4}$$

for $i = 1, 2$, where ν denotes the exterior unit normal to Γ_D. We shall refer to problem (1)–(4) as to Problem P_S, the surface problem.

This population model has received much attention since its introduction due to the interesting spatial pattern formation that its solutions may exhibit, referred to as *segregation*. Numerical experiments for the evolution problem, see [10,11,5,14], as well as analytical results on the corresponding steady state problem (with $d_i = 0$), see [17,18], seem to indicate that while the intensity of diffusion (c_i) and self-diffusion (a_{ii}) tend to suppress pattern formation, those of cross-diffusion (a_{12}, a_{21}) seem to help create segregation patterns. We refer to [26,10,11,8,5,14] and the references therein for analytical results on the existence of solutions and numerical approximations of the problem.

General competitive strategies of populations may include modifying the local environment. A good example is mangrove ecosystems [16], which are tropical communities of tree species typically growing in saline coastal soils. Mangroves are salt tolerant species which are able to exclude most of the salt from the sea-water their roots extract from the saturated soil [4]. In this way, they further salinize poorly flushed soils resulting in an increase of their comparative fitness to such areas. As pointed out by Passioura et al. [21], differences between species in strategies of water use may affect the spatial distribution of these species: species with high *transpiration rates*[1] may dominate less saline well-flushed habitats while those adapted to low transpiration rates may occupy more saline poorly flushed intertidal areas (Fig. 1).

Passioura et al. [21] provided an analytical approach to the mechanisms of soil salinization produced by mangroves and investigated the consequences of salt concentration increase on mangroves transpiration rate. Their work was later generalized and extended in a series of papers [24,12,25] from where we recall the following mathematical model. We assume the subsurface region, $\Omega \subset \mathbb{R}^N$, to be an open and bounded set that, after the introduction of dimensionless variables, see [25], takes the form $\Omega = \Gamma_D \times (0, 1)$. We denote a point in Ω by $\mathbf{x} = (\bar{x}, z)$, being z the depth. The

[1] Transpiration rate is the rate of loss of water vapor from plants surface, taking place mainly from leaves. The amount of water given off depends upon how much water the roots of the plant may absorb.