Available online at www.sciencedirect.com

ScienceDirect

Mathematics and Computers in Simulation 102 (2014) 117–130

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original Article

On some $p(x)$-quasilinear problem with right-hand side measure

E. Azroul [a], M.B. Benboubker [a,*], M. Rhoudaf [b]

[a] University of Fez, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796, Atlas Fez, Morocco

[b] Faculty of Science and Technology Ziaten, km 10, Old Airport Road, B.P. 416, Tangier, Morocco

Received 17 October 2011; received in revised form 2 September 2013; accepted 17 September 2013

Available online 25 September 2013

Abstract

In this paper we investigate the existence of entropy solution for the following nonlinear elliptic equation involving $p(x)$-Laplacian type operator,

$$-\Delta_{p(x)}u+|u|^{p(x)-2}u=\mu$$

in a bounded set $\Omega\subset\mathbb{R}^N$, coupled with a Dirichlet boundary condition. For right-hand side measure μ which admits a decomposition in $L^1(\Omega)+W^{-1,p'(x)}(\Omega)$.

Keywords: $p(x)$-Laplacian operator; Variable exponent; Entropy solutions; Truncations

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$. Our main goal is to prove the existence of entropy solution to the following nonlinear elliptic equation

$$\begin{cases}-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\mu \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega,\end{cases}\tag{1}$$

where μ is a signed measure in $L^1(\Omega)+W^{-1,p'(x)}(\Omega)$ and the variable exponent $p:\overline{\Omega}\to(1,+\infty)$ is a continuous function.

Since the operator in the divergence form is nonhomogeneous, we introduce a Sobolev space with variable exponent setting for problems of this type. On the other hand, the second term arising in the left hand side of Eq. (1) is also nonhomogeneous and its particular form appeals to a suitable variable exponent Lebesgue space setting. The study of these spaces is a new and interesting topic. One of our motivations for studying problem (1) comes from the model

* Corresponding author. Tel.: +212 666502920.
E-mail address: simo.ben@hotmail.com (M.B. Benboubker).

0378-4754/$36.00 © 2013 IMACS. Published by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.matcom.2013.09.009

proposed by Chen et al. [6] in a framework of image restoration based on a variable exponent Laplacian in which the idea behind this application requires the minimization over u of the energy

$$\int_{\Omega} |\nabla u(x)|^{p(x)} + |u(x) - I(x)|^2 \, dx, \tag{2}$$

where I is a given input, and $p(x)$ is a function varying between 1 and 2. To understand their ideas, it is useful first to look at some more classical ideas on the same problem such as the variational formulations of the isotropic and total variation smoothing.

In the isotropic smoothing one minimizes the energy

$$\int_{\Omega} |\nabla u(x)|^{p} + |u(x) - I(x)|^2 \, dx, \tag{3}$$

with $p=2$. Unfortunately, this smoothing will destroy all small details from the image, so this procedure is not very useful. A better approach is the so-called total variation smoothing. In this approach, one minimizes the energy (3) with $p=1$, so this method does a good job of preserving edges, but unfortunately, it does not preserves edges only, it also creates edges where there were none in the original image (the so-called staircase effect).

Since we would like to exploit the benefits of these two approaches, it is natural to formulate the minimization problem (3) for an exponent $p=p(x)$ varying in the interval $[1, 2]$. This is the essence of the model (2) what Chen et al. [6] did. It capitalizes on the strengths of (3) for the different values of $1 \leq p \leq 2$. It ensures TV based diffusion ($p=1$) along edges and Gaussian smoothing ($p=2$) in homogeneous regions. Furthermore, it employs anisotropic diffusion ($1<p<2$) in regions which may be piecewise smooth or in which the difference between noise and edges is difficult to distinguish.

Under our assumptions, it is reasonable to work with entropy solutions, which need less regularity than the usual weak solutions. The notion of entropy solution was proposed by Bénilan et al. [4] for nonlinear elliptic problems with constant p. Recently, Sanchön and Urbano [13] have studied the Dirichlet problem associated to the $p(x)$-Laplace equation and have obtained the existence and uniqueness of entropy solutions for L^1 data, as well as integrability results for the solution and its gradient. The proofs rely crucially on a priori estimates in Marcinkiewicz spaces with variable exponents.

We refer the reader to the papers of Bonzi and Ouaro [5] for a more general case where the authors have proved the existence and uniqueness of an entropy solution to the problem

$$\begin{cases} b(u) - \operatorname{div} a(x, \nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

where $f \in L^1(\Omega)$ and b is a continuous and nondecreasing function from $\mathbb{R}$ into $\mathbb{R}$, which is close to our problem, while $a(x, \xi) = |\xi|^{p(x)-2}\xi$ and $b(u) = |u|^{p(x)-2}u$.

The novelty of our work is to extend the second member $f \in L^1(\Omega)$ of Bonzi and Ouaro [5], and Sanchón and Urbano [13] by taking into account a signed measure $\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$. Furthermore, we prove the existence of entropy solutions by developing a method which is different from the one used in Sanchón and Urbano [13]. We proceed by approximating the initial problem for which we prove that the associated operator is pseudo-monotone. Then, by some a priori estimates, we prove that the approximate sequence converges to an entropy solution of the initial problem. The advantage of our method is that we can give the concrete expression of the limit function of the approximate solutions by means of truncation techniques and prove the strong convergence of the truncations of the approximate solutions.

The present paper is organized as follows. In Section 2, we introduce a framework for function spaces. In Section 3, we prove some fundamental lemmas concerning convergence in Sobolev spaces with variable exponent. In Section 4, we prove our results.

2. A framework for function spaces

For each open bounded subset Ω of $\mathbb{R}^N$ ($N \geq 2$), we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{p \,|\, p \in \mathcal{C}(\overline{\Omega}),\ p(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$