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Original article

Numerical approaches for state-dependent neutral delay equations with discontinuities

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Abstract

This article presents two regularization techniques for systems of state-dependent neutral delay differential equations which have a discontinuity in the derivative of the solution at the initial point. Such problems have a rich dynamics and besides classical solutions can have weak solutions in the sense of Utkin. Both of the presented techniques permit the numerical solution of such problems with the code RADAR5, which is designed to compute classical solutions of stiff and differential-algebraic (state-dependent) delay equations.

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1. Introduction

Phenomena with memory often lead to delay differential equations, and when the derivative at a time instant also depends on the derivative in the past we are concerned with neutral delay equations. In this article we are interested in systems of state-dependent neutral delay equations of the form

$$\begin{aligned} \dot y(t) &= f(y(t), \dot y(\alpha(y(t)))) \quad \text{for } t > 0 \\ y(t) &= \varphi(t) \quad \text{for } t \le 0 \end{aligned} \tag{1}$$

with smooth vector functions $f(y, z)$, $\varphi(t)$ and scalar deviating argument $\alpha(y)$ satisfying $\alpha(y(t)) < t$ (non-vanishing delay). More general equations (e.g., dependence of f on time t and on $y(\alpha(y(t)))$) could be treated as well without presenting additional difficulties. In the present article we focus on the situation, where the derivative of the solution has a jump discontinuity at the starting point, i.e.,

$$\dot\varphi(0) \ne f(\varphi(0), \dot\varphi(\alpha(\varphi(0)))). \tag{2}$$

Such a system has the following particularities:

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- since it is of neutral type, this discontinuity is in general propagated to further breaking points;
- since the deviating argument is state-dependent, it may occur that at breaking points a classical solution ceases to exist.

Let us discuss the second item in some more detail. At the first breaking point t_0, where we have $\alpha(y(t_0)) = 0$ and $\alpha(y(t)) < 0$ for $t < t_0$, the left-hand derivative of $\alpha(y(t))$ is generically positive, i.e., $\alpha'(y(t_0))f(y(t_0), \dot y(0^-)) > 0$, and we expect that the solution enters the region $\alpha(y) > 0$. However, if the right-hand derivative of $\alpha(y(t))$ is negative, i.e., $\alpha'(y(t_0))f(y(t_0), \dot y(0^+)) < 0$, it cannot enter this region, and a classical solution ceases to exist.

Such a situation is closely related to ordinary differential equations having a discontinuous vector field. In this situation it is possible to consider weak solutions (in the sense of Filippov [3] and/or Utkin [13]), where one looks for solutions staying in the manifold $\alpha(y) = 0$ (sliding mode) and one permits the derivative $\dot y(0)$ to be multi-valued.

To our knowledge, codes for delay equations cannot handle such a situation in an efficient way. Typically, the code will stop the integration at such a breaking point with the message that too small step sizes are needed. The aim of the present article is to discuss regularizations of the neutral delay equations (1), which permit the use of standard software packages for an efficient computation of classical and weak solutions.

In the present article we study two regularization techniques for the problem (1). The first one (Section 2.1) consists in changing the derivative of the initial function $\varphi(t)$ on the interval $(-\varepsilon, 0]$ in such a way that the discontinuity of $\dot y(t)$ is suppressed at the origin. The second one (Section 2.2) is based on turning the problem into the $\varepsilon \to 0$ limit of a singularly perturbed delay equation (as proposed in [7]). We shall show in Sections 3 and 4 that the solutions of the regularized problems (which are classical solutions) remain close to a solution of (1) independent of whether it is a classical or a weak one. Numerical experiments (Section 5) demonstrate the applicability of the code RADAR5 to the regularized problems, and they confirm the theoretical results of this paper.

2. Regularization techniques

The functions $f(y, z)$, $\alpha(y)$, and $\varphi(t)$ of the system (1) are assumed to be sufficiently differentiable. The discontinuity of the solution is generated by the fact that the derivative $\dot\varphi(0)$ does not match the right-hand side of the delay equation at $t = 0$. We consider two approaches (Sections 2.1 and 2.2) of regularizing this discontinuity. Other regularizations have been considered in [4], where the right-hand side is replaced by its average on a small interval, and in [1], where the problem is regularized by its numerical discretization based on the Euler method (an idea also used for other classes of differential equations as discussed in [2]).

The analysis of singularly perturbed state-dependent delay equations is an interesting subject in itself and has received the attention of many researchers in recent years (see e.g. [11]) both from the theoretical and the numerical point of view.

2.1. Regularization of the initial function

By introducing a new variable for the derivative, a neutral delay equation can be transformed into a differential-algebraic delay equation. In our situation Eq. (1) becomes

$$\begin{aligned} \dot y(t) &= z(t) \\ 0 &= f(y(t), z(\alpha(y(t)))) - z(t), \end{aligned} \tag{3}$$

where $y(t) = \varphi(t)$ and $z(t) = \dot\varphi(t)$ for $t \le 0$. This permits us to treat the functions $y(t)$ and $z(t)$ independently of each other. We do not touch the condition $y(t) = \varphi(t)$ for $t \le 0$, but we replace the condition $z(t) = \dot\varphi(t)$ on the interval $-\varepsilon \le t \le 0$ by

$$z(t) = \dot\varphi(-\varepsilon) + \chi\left(\frac{t}{\varepsilon}\right)(\dot y_0^+ - \dot\varphi(-\varepsilon)), \qquad \dot y_0^+ = f(\varphi(0), \dot\varphi(\alpha(\varphi(0)))), \tag{4}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function satisfying $\chi(-1) = 0$, $\chi(0) = 1$, and $\chi'(\tau) > 0$ for $\tau \in [0, 1]$, e.g., the linear interpolation polynomial $\chi(\tau) = \tau + 1$. In this way, the function $z(t)$ is continuous at $t = 0$ and the problem will have a (classical) solution, where the original problem did not. Consequently, codes for differential-algebraic (index 1), state-dependent delay equations can be applied to solve the problem.