Available online at www.sciencedirect.com
SciVerse ScienceDirect
Mathematics and Computers in Simulation 95 (2013) 39–54
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original article

Bifurcation of discontinuous limit cycles of the Van der Pol equation

Marat Akhmet[a,*], Mehmet Turan[b]

[a] Department of Mathematics, Middle East Technical University, 06800 Ankara, Turkey
[b] Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey

Received 30 August 2011; received in revised form 9 October 2012; accepted 5 May 2013
Available online 14 May 2013

Abstract

In this paper, we apply the methods of B-equivalence and ψ-substitution to prove the existence of discontinuous limit cycle for the Van der Pol equation with impacts on surfaces. The result is extended through the center manifold theory for coupled oscillators. The main novelty of the result is that the surfaces, where the jumps occur, are not flat. Examples and simulations are provided to demonstrate the theoretical results as well as application opportunities.

MSC: 34A37; 34C23; 34C25; 37G15; 37N20

Keywords: Discontinuous limit cycle; Van der Pol equation; Hopf bifurcation; Coupled oscillators; Center manifold

1. Introduction and preliminaries

Getting bifurcation in dynamics with impacts relies mainly on collisions near the impact point(s). That is why they are called corner-collision, border-collision, crossing-sliding, grazing-sliding, switching-sliding, etc., bifurcations [9,12,13,15,17,22,23,30]. That is, the bifurcations are located geometrically. In our present result, we do not have the geometrical source of bifurcation. It is rather reasoned by specifically arranged interaction of continuous and discontinuous stages of the process. To be precise, we use a generalized eigenvalue to evaluate which we apply a characteristic of the impact as well as of the continuous process between moments of discontinuity. This approach when continuous and discontinuous stages are equally participated in creating a certain phenomena is common for the theory of differential equations with impulses [3,34]. Our results are, rather, close to those, which obtained for systems where continuous flows and surfaces of discontinuity are transversal [2,3,5,14,24].

The main instruments in our paper, except for the Hopf bifurcation technique, are the methods of B-equivalence and ψ-substitution developed in our papers [1–4,6] for discontinuous limit cycles, and one has to emphasize that the set of all periodic solutions of the non-perturbed system is a proper subset of all solutions near the origin. By a discontinuous cycle, we mean a trajectory of a discontinuous periodic solution.

The Van der Pol equation arises in the study of circuits containing vacuum tubes and is given by

$$y'' + \varepsilon(1 - y^2)y' + y = 0 \quad (1)$$

* Corresponding author. Tel.: +90 312 210 53 55; fax: +90 312 210 29 72.
E-mail addresses: marat@metu.edu.tr (M. Akhmet), mturan@atilim.edu.tr, mehmetturan21@gmail.com (M. Turan).
URLs: http://www.metu.edu.tr/~marat/ (M. Akhmet), http://www.atilim.edu.tr/~mturan/ (M. Turan).

0378-4754/$36.00
http://dx.doi.org/10.1016/j.matcom.2013.05.002

where ε is a real parameter. If $\varepsilon = 0$, the equation reduces to the equation of simple harmonic motion $y'' + y = 0$. The term $\varepsilon(1-y^2)y'$ in (1) is usually regarded as the friction or resistance. If the coefficient $\varepsilon(1-y^2)$ is positive, then we have the case of "positive resistance", and when the coefficient $\varepsilon(1-y^2)$ is negative then we have the case of "negative resistance". This equation, introduced by Lord Rayleigh (1896), was studied by Van der Pol (1927) [36] both theoretically and experimentally using electric circuits.

Hopf bifurcation is an attractive subject of analysis for mathematicians as well as for mechanics and engineers [2,5,8,11,13–15,22,25,26,28,31,35,36]. Many papers and books have been published about mechanical and electrical systems with impacts [7,9,12,17,20,27,30,37].

We consider the model with impulses on surfaces which are places in the phase space and are essentially nonlinear while it is known that the Hopf bifurcation is considered either with linear surfaces of discontinuity or with fixed moments of impulses [7–10,13,15,17,18,22,33]. We have developed a special effective approach to analyze the problem in depth which consists of the method of reduction of equations with variable moments of impacts to systems with fixed moments of impacts [3], a class of equations on variable time scales [2,6], a transformation of equations on time scales to systems with impulses [4]. This is all the theoretical basis of the present results.

Specifically, we consider the following system:

$$\begin{aligned} &y'' + 2\alpha y' + (\alpha^2+\beta^2)y = F(y, y', \mu), \quad (y, y') \notin \Gamma(\mu),\\ &\Delta y'|_{(y,y')\in\Gamma(\mu)} = cy + dy' + J(y, y', \mu), \end{aligned} \tag{2}$$

where α, $\beta \neq 0$, c, d are real constants with $c = \alpha d$, F and J are analytic functions in all variables. $\Gamma(\mu)$ is the set of discontinuity whose equation is given by $m_1 y + m_2 y' + \tau(y, y', \mu) = 0$, $y > 0$, for some real numbers m_1, m_2, and the function $\tau(y, y', \mu)$ stands for a small perturbation, $\Delta y'|_{(y,y')\in\Gamma(\mu)} = y'(\theta^+) - y'(\theta)$ denotes the jump operator in which θ is the time when the solution (y, y') meets the discontinuity set $\Gamma(\mu)$, that is, θ is such that $m_1 y(\theta) + m_2 y'(\theta) + \tau(y(\theta), y'(\theta), \mu) = 0$, and $y'(\theta^+)$ is the right limit of $y'(t)$ at $t = \theta$. After the impact, the phase point $(y(\theta^+), y'(\theta^+))$ will belong to the set $\Gamma'(\mu) = \{(u, v) \in \mathbb{R}^2 : u = y, v = cy + (1+d)y' + J(y, y', \mu), (y, y') \in \Gamma(\mu)\}$. Here $y(\theta^+)$ is the right limit of $y(t)$ at $t = \theta$. One can easily see that nonlinearity is inserted into all parts of the model including the surface of discontinuity.

If we choose $\alpha = \varepsilon/2$, $\beta = \sqrt{1-\alpha^2}$ and $F(y, y', \mu) = \varepsilon y^2 y'$ in the differential equation of the system (2), then the *Van der Pol equation* will be obtained. Therefore, (1) is a special case of (2), if the impulsive condition is not considered. Note that if $F(y, y', \mu) = \varepsilon_2 y^2 y'$ for some nonzero constant ε_2, we still have (1) after using the linear transformation $y = \sqrt{\varepsilon/\varepsilon_2}\, z$ of the dependent variable.

To explain our application motivations, we consider the oscillator which is subdued to the impacts modeled by the Newton's law of restitution as a concrete mechanical problem. Consider the system

$$\begin{aligned} &y'' + \varepsilon_1 y' + y = \varepsilon_2 y^2 y', \quad (y, y') \notin \Gamma,\\ &\Delta y'|_{(y,y')\in\Gamma} = dy', \end{aligned} \tag{3}$$

where ε_1, ε_2 are constants, $d = e^{2\pi\varepsilon_1(4-\varepsilon_1^2)^{-1/2}} - 1$, Γ is the half line $y = 0$, $y' > 0$. As mentioned above, the last system is a generalization of the Van der Pol equation with impacts of Newton's type. If one takes (3) with $\varepsilon_2 = 0$, then the system is

$$\begin{aligned} &y'' + \varepsilon_1 y' + y = 0, \quad (y, y') \notin \Gamma,\\ &\Delta y'|_{(y,y')\in\Gamma} = dy'. \end{aligned} \tag{4}$$

Note that the general solution of the differential equation without impulse condition in (4) is given by

$$y(t) = e^{-\varepsilon_1 t/2}\left(C_1 \cos\left(\frac{(4-\varepsilon_1^2)^{1/2} t}{2}\right) + C_2 \sin((4-\varepsilon_1^2)^{1/2} t/2)\right), \tag{5}$$

where C_1 and C_2 are arbitrary real constants. Let $(0, y_0')$ be any point on the line $\Gamma' = \Gamma$. That is, assume that $y_0' > 0$. Then $y(0) = 0$, $y'(0) = y_0'$ in (5) gives us $C_1 = 0$, $C_2 = 2y_0'(4-\varepsilon_1^2)^{-1/2}$. Thus, we obtain

$$y(t) = 2y_0'(4-\varepsilon_1^2)^{-1/2} e^{-\varepsilon_1 t/2} \sin((4-\varepsilon_1^2)^{1/2} t/2).$$