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Original article

Timestepping schemes for nonsmooth dynamics based on discontinuous Galerkin methods: Definition and outlook

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Abstract

The contribution deals with timestepping schemes for nonsmooth dynamical systems. Traditionally, these schemes are locally of integration order one, both in non-impulsive and impulsive periods. This is inefficient for applications with infinitely many events but large non-impulsive phases like circuit breakers, valve trains or slider-crank mechanisms. To improve the behaviour during non-impulsive episodes, we start activities twofold. First, we include the classic schemes in time discontinuous Galerkin methods. Second, we split non-impulsive and impulsive force propagation. The correct mathematical setting is established with mollifier functions, Clenshaw–Curtis quadrature rules and an appropriate impact representation. The result is a Petrov–Galerkin distributional differential inclusion. It defines two Runge–Kutta collocation families and enables higher integration order during non-impulsive transition phases. As the framework contains the classic Moreau–Jean timestepping schemes for constant ansatz and test functions on velocity level, it can be considered as a consistent enhancement. An experimental convergence analysis with the bouncing ball example illustrates the capabilities.

Keywords: Timestepping scheme; High order; Nonsmooth dynamics; Time discontinuous Galerkin methods; Impact

Notation

The following notation is used throughout the paper. Let I denote a real time interval. A function $f : I \to \mathbb{R}^n$ is said to be of class $\mathcal{C}^p(I;\mathbb{R}^n)$ if it is continuously differentiable up to the order p. The set of functions $f : I \to \mathbb{R}^n$ that are absolutely continuous on I is denoted by $\mathcal{W}^{1,1}(I;\mathbb{R}^n)$. The set of functions $f : I \to \mathbb{R}^n$ that are locally Lebesgue integrable on I is referred to as $L^1_{\text{loc}}(I;\mathbb{R}^n)$. The set of functions $f : I \to \mathbb{R}^n$ of bounded variations (BV) is represented by $\mathcal{BV}(I;\mathbb{R}^n)$. For $f \in \mathcal{BV}(I;\mathbb{R}^n)$, the right-limit function is given by $f^+(t) = \lim_{s\to t, s>t} f(s)$, and respectively the left-limit function by $f^-(t) = \lim_{s\to t, s<t} f(s)$. The jump of f at t is symbolized by $[[f(t)]] = f^+(t) - f^-(t)$. The set of functions $f : I \to \mathbb{R}^n$ of locally bounded variations (LBV) is expressed as $\mathcal{LBV}(I;\mathbb{R}^n)$. In all cases, we skip the image space if there is no ambiguity and we extend the domain if necessary.

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The set of measures on the interval I is represented by $\mathcal{M}(I)$. We associate with any function $f \in \mathcal{LBV}(I)$ a differential measure $\mathrm{d}f \in \mathcal{M}(I)$ [18]. The notation $\mathrm{d}t$ defines the Lebesgue measure on $\mathbb{R}$. The space of all real-valued, $\mathcal{C}^\infty$-functions with compact support in I is denoted by $\mathcal{D}(I)$. The set of linear functionals that maps $\mathcal{D}(I)$ onto the set of real numbers defines the dual space $\mathcal{D}^*(I)$, which is called the space of distributions. For a distribution $d \in \mathcal{D}^*(I)$, it is conventional to write

$$d : \mathcal{D}(I) \to \mathbb{R}, \quad \varphi \mapsto \langle d, \varphi \rangle \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the primal-dual pairing and $\langle d, \cdot \rangle$ is the linear functional which defines d. For $f \in L^1_{\mathrm{loc}}(I; \mathbb{R}^n)$ (respectively a measure $\mu \in \mathcal{M}(I)$), a corresponding distribution T_f (respectively T_μ) is associated such that

$$\langle T_f, \varphi \rangle = \int_I f\varphi \mathrm{d}t \quad \left(\text{respectively } \langle T_\mu, \varphi \rangle = \int_I \varphi\mu \right). \tag{2}$$

One abuses notation by identifying T_f with f, i.e. $\langle f, \varphi \rangle = \langle T_f, \varphi \rangle$ (respectively T_μ with μ, $\langle \mu, \varphi \rangle = \langle T_\mu, \varphi \rangle$). The distributional derivative of a distribution d will be symbolized by Dd and is usually defined by

$$\langle Dd, \varphi \rangle := -\langle d, \dot{\varphi} \rangle, \quad \forall \varphi \in \mathcal{D}(I). \tag{3}$$

We denote by $0 =: t_0 < t_1 < \cdots < t_k < \cdots < t_N := T$ a finite partition (or a subdivision) of the time interval $[0, T]$ $(T > 0)$. The integer N stands for the number of time intervals in the subdivision. The N sub-intervals $I_i := (t_{i-1}, t_i)$ are of length Δt_i and define the time-steps. The time step-size partition is referred to as $\mathcal{I} := \{I_1, \ldots, I_N\}$. The set of piecewise continuously differentiable functions on this subdivision is given by $\mathcal{C}^p(\mathcal{I}; \mathbb{R}^n)$. The value of a real function $x(t)$ at the time t_k is approximated by x_k.

1. Point of departure

This article treats higher order timestepping schemes based on time discontinuous Galerkin methods in the context of nonsmooth dynamics. We give a short introduction of nonsmooth dynamical systems in mechanics, of classical time integration schemes and of present strategies to achieve higher integration order during non-impulsive episodes.

1.1. Nonsmooth dynamical systems

The *bouncing ball* (cf. Fig. 1) is a typical *nonsmooth dynamical system* in the field of mechanics [29,10,6,24,16,2,26]. Informally, we can envisage the physical evolution as follows. During a finite time interval $\emptyset \neq I := (0, T) \subset \mathbb{R}$, a ball with mass m falls from an initial position q_0, given an initial velocity v_0 and some external *momentum flow* $f\mathrm{d}t$. It hits the ground and lifts off again or stays calm depending on the resulting interaction $\mathrm{d}i$ being partly elastic or plastic. If the impact events accumulate in finite-time, the first case is called a *Zeno phenomenon* if bouncing and free flight alternate infinitely often in I.

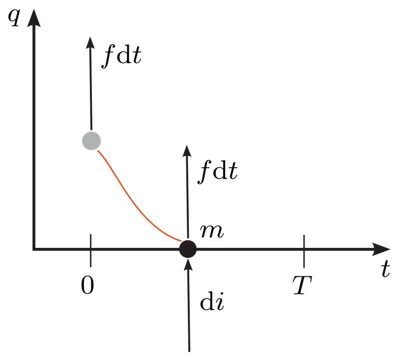

Fig. 1. Bouncing ball example.