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Original article

On some generalizations of the implicit Euler method for discontinuous fractional differential equations☆

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Abstract

We discuss the numerical solution of differential equations of fractional order with discontinuous right-hand side. Problems of this kind arise, for instance, in sliding mode control. After applying a set-valued regularization, the behavior of some generalizations of the implicit Euler method is investigated. We show that the scheme in the family of fractional Adams methods possesses the same chattering-free property of the implicit Euler method in the integer case. A test problem is considered to discuss in details some implementation issues and numerical experiments are presented.

MSC: 65L06; 34A08; 34A36; 47N70

Keywords: Fractional differential equation; Discontinuous problem; Set-valued regularization; Implicit method; Fractional Adams method

1. Introduction

The term fractional calculus refers to the generalization of integration and differentiation to any arbitrary (i.e., non necessarily integer) order. This idea is by no means new and was pioneered by Leibniz who mentioned the possibility of derivatives of order 1/2 in a correspondence exchanged with L'Hospital in 1695; successively, fractional calculus stimulated many famous mathematicians, including Euler, Fourier, Lagrange, Laplace, Riemann and some others.

Despite this long history, relevant applications of fractional calculus have emerged only a few decades ago; nonetheless, models of fractional order are nowadays commonly used in several areas, ranging from chemistry and physics to biology, engineering, finance and so on. See [34] for an historical survey of fractional calculus.

In control theory it has been observed that the introduction of controllers involving integration or derivation of non-integer order allows to achieve higher performance with respect to classical systems of integer order; we refer the reader to one of the recent monographes [6,33,35,37] on fractional systems.

Sliding mode control (SMC) is a special class of variable-structure systems; it is designed to alter the dynamics of the system which is firstly driven toward a switching surface and hence is constrained to stay on it. The success and

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widespread use of SMC is mainly due to its simplicity and robustness against parameter variations and disturbances [42].

Very recently the investigation of the effects of SMC techniques on fractional systems has been approached (e.g., see [4,18,36,39,38]). Since in SMC the control law is not a continuous function but switches from one continuous structure to another (according to the position of the system in the portion of the state delimited by the switching surface), its use for fractional order systems poses new and significant challenges, especially for the numerical computation.

In [2] the behavior of the implicit Euler (IE) method for the numerical simulation of non-smooth dynamical systems of integer order has been analyzed. It has been showed that, unlike the explicit Euler method which generates unwanted spurious oscillations, the implicit scheme allows a smooth stabilization on the switching surface. This important feature, which has been achieved after recasting the system into a Filippov's differential inclusion framework, validates the IE as a viable method for chattering suppression.

The main aim of this paper is to introduce the study of the counterparts of the IE method for fractional differential equations (FDEs) when applied to problems with discontinuous right-hand side. As it is known, the IE method can be generalized to FDEs according to different approaches, which give rise to different methods: we intend to verify whether implicit schemes are able to prevent chattering phenomena also in the fractional case and detect which of the generalizations of the IE method possess this feature. Furthermore we intend to discuss some of the major issues for the implementation of implicit methods in the context under investigation.

This paper is organized as follows. In Section 2 some basic facts on fractional calculus are reviewed, FDEs with discontinuity are introduced and some results concerning the Filippov's regularization of discontinuous FDEs are discussed. In Section 3 we present some of the most used numerical methods for FDEs. Interestingly, their application to a discontinuous test FDE in Section 4 discloses some unexpected features: different schemes leading to the same method when the order α of the FDE tends to the nearest integer (i.e., when the FDE tends to an ordinary differential equation (ODE)) behave in a different way in the presence of discontinuities; furthermore, only the method belonging to the class of fractional Adams methods seems to preserve the chattering-free motion observed for the IE method in the integer case. In Section 5 the attention is moved to the more general test problem introduced in [39] and we discuss some practical aspects related to the use of implicit methods. Finally, in Section 6 we present the results of some numerical simulations.

2. Differential inclusions of fractional order

Historically, the origins of fractional calculus are strictly related to the *Riemann–Liouville* definition of the integral of order $\alpha > 0$ on the interval $[t_0, t]$

$$J_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is the Euler gamma function (for references to introductory material on fractional calculus the reader is referred to any classical textbook on the subject, for instance [15,28,31,40]).

The definition of differential operators of fractional order is not unique and different approaches have been proposed. For instance, the *Riemann–Liouville* (RL) differential operator of order α is defined as

$${}^{RL}D_{t_0}^{\alpha} f(t) \equiv D^m J_{t_0}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} (t-s)^{m-\alpha-1} f(s) ds,$$

where $m = \lceil \alpha \rceil$ is the smallest integer such that $m > \alpha$ and D^m and d^m/dt^m denote the standard derivative of integer order.

An alternative definition, commonly named as the *Caputo* differential operator, has been introduced in [7,8] and it is defined according to

$${}^{C}D_{t_0}^{\alpha} f(t) \equiv J_{t_0}^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s) ds.$$