Available online at www.sciencedirect.com

SciVerse ScienceDirect

Mathematics and Computers in Simulation 127 (2016) 56–79

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original article

# Numerical approximation of solitary waves of the Benjamin equation

V.A. Dougalis [a,b], A. Durán [c,d,*], D.E. Mitsotakis [e]

[a] Mathematics Department, University of Athens, 15784 Zographou, Greece
[b] Institute of Applied and Computational Mathematics, F.O.R.T.H. 71110, Heraklion, Greece
[c] Department of Applied Mathematics, University of Valladolid, Campus Miguel Delibes, P/Belen 15 E-47011 Valladolid, Spain
[d] IMUVA, Instituto de Investigación en Matemáticas, University of Valladolid, Spain
[e] Institute for Mathematics and its Applications, University of Minnesota, 114 Lind Hall, 207 Church Street SE, Minneapolis, MN 55455, United States

Received 20 October 2011; received in revised form 18 April 2012; accepted 10 July 2012
Available online 2 August 2012

**Abstract**

This paper presents several numerical techniques to generate solitary-wave profiles of the Benjamin equation. The formulation and implementation of the methods focus on some specific points of the problem: on the one hand, the approximation of the nonlocal term is accomplished by Fourier techniques, which determine the spatial discretization used in the experiments. On the other hand, in the numerical continuation procedure suggested by the derivation of the model and already discussed in the literature, several algorithms for solving the nonlinear systems are described and implemented: the Petviashvili method, the Preconditioned Conjugate Gradient Newton method and two Squared-Operator methods. A comparative study of these algorithms is made in the case of the Benjamin equation; Newton's method combined with Preconditioned Conjugate Gradient techniques, emerges as the most efficient. The resulting numerical profiles are shown to have a high accuracy as travelling-wave solutions when they are used as initial conditions in a time-stepping procedure for the Benjamin equation. The paper also explores the generation of multi-pulse solitary waves.


*Keywords:* Benjamin equation; Numerical computation of solitary waves; Numerical continuation; Spectral methods

## 1. Introduction

The goal of this paper is the description and comparison of some techniques to compute solitary-wave profiles of the Benjamin equation

$$u_t + \alpha u_x + \beta u u_x - \gamma \mathrm{H} u_x - \delta u_{xxx} = 0, \tag{1}$$

* Corresponding author at: Department of Applied Mathematics, University of Valladolid, Campus Miguel Delibes, P/Belen 15 E-47011 Valladolid, Spain.

*E-mail addresses:* doug@math.uoa.gr, dougalis@iacm.forth.gr (V.A. Dougalis), angel@mac.uva.es (A. Durán), dmitsot@gmail.com (D.E. Mitsotakis).

0378-4754/$36.00 
http://dx.doi.org/10.1016/j.matcom.2012.07.008

In (1), $\alpha$, $\beta$, $\gamma$, $\delta$ are nonnegative constants and $H = \mathcal{H}\partial_x$, where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

The Fourier symbol of the operator H is

$$\widehat{(\mathrm{H}u)}(\xi) = |\xi|\widehat{u}(\xi).$$

Particular cases of (1) are the Korteweg–de Vries (KdV) equation ($\gamma = 0$, $\delta > 0$) and the Benjamin–Ono (BO) equation ($\gamma > 0$, $\delta = 0$). Eq. (1) arises in the study of unidirectional propagation of long internal waves of small amplitude at the interface of two ideal fluids (the heavier of which has infinite depth) in the presence of surface tension [5–7,2]. Global well-posedness in $L^2$ for the corresponding initial-value problem is proved in [25] (where other results concerning generalized versions of (1) are also obtained). It is well known that the functionals [6]

$$C(u) = \int_{-\infty}^{\infty} u\, dx, \tag{2}$$

$$I(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2\, dx, \tag{3}$$

$$E(u) = \alpha I(u) + \int_{-\infty}^{\infty} \left(\frac{\beta}{6}u^3 - \frac{1}{2}\gamma u \mathrm{H}u + \frac{1}{2}\delta u_x^2\right) dx, \tag{4}$$

are preserved by sufficiently smooth, suitably vanishing at infinity solutions of (1). The quantity (4) determines the Hamiltonian formulation of (1)

$$u_t = J\delta E(u), \quad J = -\partial_x.$$

in suitable function spaces and where $\delta E$ denotes the variational derivative.

Solitary-wave solutions $u(x, t) = \varphi(x - c_s t)$, $c_s > 0$ of (1) were initially studied, in terms of existence, stability and asymptotic behaviour, by Benjamin in [5–7]. The profile $\varphi$ satisfies, assuming that $\varphi \to 0$ as $|x| \to \infty$, the equation

$$F(\varphi, \gamma) = \delta E(\varphi) - c_s \delta I(\varphi) = (-c_s + \alpha)\varphi + \frac{\beta}{2}\varphi^2 - \gamma \mathrm{H}\varphi - \delta\varphi'' = 0. \tag{5}$$

Benjamin argued that (5) has solutions that exhibit an oscillating behaviour in some parts of the spatial domain and a monotonic decay at infinity as $1/|x|^2$. A complete theory of existence and stability of solitary-wave solutions for small values of $\gamma$ is provided in [2].

Explicit formulas for the profiles (except in the KdV and BO cases) are not known. Several techniques of numerical approximation involve numerical continuation algorithms on some parameter in (1). Numerical continuation is applied in [2] to design a code to approximate the solitary waves. The results reveal some properties, such as the symmetry about their crests and (except in the KdV and BO cases) the existence of a finite number of oscillations. In addition, numerical studies in [22] suggest an inelastic interaction of solitary waves and consequently a nonintegrable character of (1) when $\gamma > 0$, $\delta > 0$.

Note that if we look for solutions of (5) in the form

$$\varphi(X) = -\frac{2(\alpha - c_s)}{\beta}\psi\left(\sqrt{\frac{\alpha - c_s}{\delta}}X\right), \tag{6}$$

then the profile $\psi$ satisfies

$$Q(\psi, \tilde{\gamma}) = \psi - 2\tilde{\gamma}\mathrm{H}\psi - \psi'' - \psi^2 = 0, \tag{7}$$

where

$$\tilde{\gamma} = \frac{\gamma}{2\sqrt{\delta(\alpha - c_s)}}. \tag{8}$$