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Mathematics and Computers in Simulation 127 (2016) 94–100

MATHEMATICS AND COMPUTERS IN SIMULATION

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Original Articles

Hyperbolic diffusion with Christov–Morro theory

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Received 23 September 2011; received in revised form 7 June 2012; accepted 5 July 2012
Available online 31 July 2012

Abstract

We employ recent ideas of C.I. Christov and of A. Morro to develop a theory for diffusion of a solute in a Darcy porous medium taking convection effects into account. The key point is that the solute evolution is not governed by a parabolic system of equations. Indeed, the theory developed is basically hyperbolic. This still leads to a model which allows for convective (gravitational) overturning in a porous layer, but in addition to the classical mode of stationary convection instability there is the possibility of oscillating convection being dominant for a lower salt Rayleigh number, if the relaxation time is sufficiently large.

Keywords: Convection in porous media; Hyperbolic diffusion; Christov concentration flux equation; Linear instability; Oscillatory convection

1. Introduction

The problem of movement of a solute in a porous medium is one with many applications in pollution, transport of radio-active waste, and many other things. There are many models for movement of a solute but typically these are based on Fick's law and lead to a parabolic equation, or a system of such equations, cf. Graffi [16], Prouse and Zaretti [26], Franchi and Straughan [12], Straughan [28]. Also, there are many detailed numerical analyses and simulations of solute transport in such systems, see e.g. Ewing et al. [7–9].

Recently, there has been a lot of interest in developing and analysing models for solute transport which have hyperbolic characteristics rather than parabolic. For example, the early work of Galenko and Danilov [13], Sobolev [27], and a very useful review may be found in Galenko and Jou [14]. A parallel mathematical analysis development has also taken place, see e.g. Gatti et al. [15], Bonetti et al. [1], Grasselli et al. [17], Wu et al. [32], and a good review of the mathematical literature is contained in Jiang [20], see also Section 9 of Straughan [31]. The development of hyperbolic transport equations follows the analogous hyperbolic heat propagation theory which is analysed in the articles of Christov and Jordan [3], Christov and Jordan [6], Jordan [22], Jordan et al. [21] and the many references therein.

There are many novel applications of a hyperbolic theory of solute transport. For example, we mention material transfer in stars, Herrera and Falcón [19], Falcón [10], Straughan [30], drug delivery in the human body, Ferreira and de Oliveira [11], and other biological and medical applications are considered in chapter 9 of the book by Straughan [31]. Thus, we believe there is strong motivation to develop and analyse a hyperbolic theory of convective overturning

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0378-4754/$36.00
http://dx.doi.org/10.1016/j.matcom.2012.07.010

of a solute in a layer of a porous material. To do this we employ recent work of Christov [4,5] who advocate the use of a Lie derivative and Morro [23,24] who develops a thermodynamically consistent theory which is compatible with Christov's derivative.

2. The model

Let $\hat{\phi}$ be the porosity in a porous medium, cf. Straughan [28], p. 1, and let V_i be the actual velocity of fluid in the pores of a saturated porous medium. We here take the porosity to be constant. Denote by $v_i = \hat{\phi} V_i$ the pore averaged velocity. Then, for a solute dissolved in the fluid, with concentration $C(\mathbf{x}, t)$ Darcy's law governing the velocity field is, cf. Straughan [28], pp. 10–12,

$$0 = -\frac{\partial p}{\partial x_i} - \frac{\mu}{k} v_i - \rho_0 g k_i \alpha C \,, \tag{1}$$

where p, μ, k, ρ_0, g, α are the pore averaged pressure, dynamic viscosity of the fluid, the permeability, the constant density (employing a Boussinesq approximation), gravity, and the coefficient of salt dependence in the density law. Here $\mathbf{k} = (0, 0, 1)$ and standard indicial notation will be employed throughout. We have neglected the acceleration in Eq. (1), since this is believed negligible in a Darcy porous material, cf. Straughan [28]. In addition the fluid is incompressible, so

$$v_{i,i} = 0 \,. \tag{2}$$

To describe solute movement in the pores we suppose the solute concentration satisfies the equations

$$\frac{\partial C}{\partial t} + V_i \frac{\partial C}{\partial x_i} = -\frac{\partial J_i}{\partial x_i} \,, \tag{3}$$

and

$$\begin{aligned} &\tau \left(\frac{\partial J_i}{\partial t} + V_j \frac{\partial J_i}{\partial x_j} - J_j \frac{\partial V_i}{\partial x_j} \right) = \\ &-k_c \frac{\partial C}{\partial x_i} - J_i + \xi_1 \Delta J_i + \xi_2 J_{k,ki} \,. \end{aligned} \tag{4}$$

In these equations $\mathbf{J}$ is the solute flux, Eq. (3) expresses concentration balance, and Eq. (4) is a generalization of Fick's law. The parameter $\tau(>0)$ is a relaxation time, k_c is a coefficient of solute diffusion, the derivative on the left in Eq. (4) is proposed by Christov [4], ξ_1, $\xi_2 > 0$ are coefficients corresponding to terms introduced by Morro [23]. (We remark that when $\tau = 0$ and $\xi_1 = \xi_2 = 0$, Eqs. (3) and (4) reduce to the classical equation of Fick's law and parabolic transport of solute.)

In keeping with Eq. (1), it is convenient to rewrite Eqs. (3) and (4) in terms of the pore averaged velocity v_i so that we have

$$\hat{\phi} \frac{\partial C}{\partial t} + v_i C_{,i} = -\hat{\phi} J_{i,i} \,, \tag{5}$$

$$\begin{aligned} &\tau \left(\hat{\phi} \frac{\partial J_i}{\partial t} + v_j \frac{\partial J_i}{\partial x_j} - J_j \frac{\partial v_i}{\partial x_j} \right) = \\ &-\hat{\phi} J_i - \hat{\phi} k_c C_{,i} + \hat{\phi} \xi_1 \Delta J_i + \hat{\phi} \xi_2 J_{k,ki} \,. \end{aligned} \tag{6}$$

The complete model for solute transport in a porous material consists of equations (1),(2),(5) and (6), and is a system in the variables v_i, p, C and J_i .