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Original article

On the dilatational wave motion in anisotropic fractal solids

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Abstract

This paper reports a study of wave motion in a generally anisotropic fractal medium (i.e. with different fractal dimensions in different directions), whose constitutive response is represented by an isotropic Hooke's law. First, the governing elastodynamic laws are formulated on the basis of dimensional regularization. It is discovered that the satisfaction of the angular momentum equation precludes the implementation of the classical elasticity theory which results in symmetry of the Cauchy stress tensor. Nevertheless, the classical elastic constitutive model can still be applied to explore *dilatational* wave propagation; in such a case, the angular momentum balance is "trivially" satisfied. The resulting problem, of eigenvalue type, is solved analytically. A computational finite element method solver is also developed to simulate the problem in its one-dimensional (1d) 1D form, it is validated by the reference solutions generated through the modal analysis. A 3D finite-difference-based solver is also developed and the obtained results match those of the 1D simulation.

Keywords: Elastodynamics; Fractal solid; Dilatational wave

1. Introduction

Fractal mechanics is a current area of research in science and engineering. Fractal media are abundant in nature (rocks, tree leaves) and in living bodies (brain). However, the conventional and well-known continuum mechanics lacks a modelling capability for such materials, and that is the issue motivating the growing activity in the so-called *fractal mechanics* [5]. Materials which exhibit a highly irregular (non-smooth) topology or inherent porosity are often treated in the fractal framework where a dimensional regularization (homogenization technique) is applied to map their fractal geometry to an equivalent Euclidean one. In general, fractal materials are of two types: random (natural) or regular (geometric); the latter exhibits a perfectly defined fractality, i.e. its Hausdorff dimension is mathematically determined via the self-similarity property applied in its construction [9], while the former has a statistical fractality. The development of a field theory, in which the fundamental laws of mechanics (conservation of mass, linear and angular momentum) are reproduced in the fractal setting resulting in continuum-type equations, is the current focus of many scientists [18,14].

Some fractal media can be classified as isotropic, but in general, fractals are anisotropic, i.e. their Hausdorff dimension is direction-dependent. Wave propagation in isotropic fractal media on spherical domains has been studied in [11,18], whereas a formulation of a three-dimensional (3D) elastodynamic model for anisotropic fractal solids was

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developed in [14] based on product measures. The latter work utilized two different approaches to derive the governing elastodynamic equations of this material model: the fractal mechanics approach and the variational energy approach with both approaches producing consistent results. This attribute is not realized in previous fractal analysis research where a single approach is followed to describe the mechanics of the model. In brief, Li and Ostoja-Starzewski [14] adopted the modified Riemann-Liouville fractional integral formulated by Jumarie [13] to derive the mass power law, Gauss's law and the transport theorems in fractal domains. Accordingly, the fractional integral of a certain function $f(x)$ over a fractal set of Hausdorff dimension $D \in]0, 1]$, with support on $[0, L]$, is regularized as follows

$$\int_0^L f(x)(\mathrm{d}x)^D = \int_0^L D(L-x)^{D-1} f(x)\mathrm{d}x \tag{1}$$

In this regard, the mass power law of a 3D fractal body (V_D) embedded in the Cartesian system $Ox_1x_2x_3$ is derived as follows

$$\begin{aligned} m &= \int_{V_D} \rho(x_i)\mathrm{d}V_D = \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \rho(x_1,x_2,x_3)(\mathrm{d}x_3)^{D_3}(\mathrm{d}x_2)^{D_2}(\mathrm{d}x_1)^{D_1} \\ &= \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \rho(x_1,x_2,x_3)c_1c_2c_3\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 = \int_V \rho(x_i)c_V\mathrm{d}V \end{aligned} \tag{2}$$

where ρ is the local mass density of the body and c_i is the product measure along the ith direction defined as

$$c_i = D_i(L_i - x_i)^{D_i - 1} \quad i = 1, 2, 3 \tag{3}$$

and c_V is a volumetric product measure defined as the product of all three product measures, thus $c_V = c_1c_2c_3$. Note that for a uniform density $\rho \equiv \rho_0$, the total mass becomes $m = \rho_0 L_1^{D_1} L_2^{D_2} L_3^{D_3}$. This finding is mathematically consistent with the universal mass power law of a fractal body with characteristic length L and Hausdorff dimension D, $m \sim L^D$ [18]. In conclusion, c_i is applied to regularize 1D or length integrals, while its equivalent, c_V, regularizes 3D or volume integrals. Consistent with this, a surface product measure is introduced to regularize surface integrals as will be later shown in deriving the Green–Gauss theorem on fractal media.

At this point, we discuss the fractional reproduction of the Green–Gauss theorem, a widely applied integral identity in formulating many continuum mechanics principles. This analysis will result in defining the mathematical form of the fractal derivative, which will be utilized in generalizing the fundamental balance laws for mechanics in fractal media. In continuous domains, the Green–Gauss theorem correlates the integral of a vector function f_i over a closed surface ∂W, to the integral of its divergence over the volume W enclosed by this surface [17], thus

$$\int_{\partial W} f_i n_i \mathrm{d}S = \int_W f_{i,i}\mathrm{d}V \tag{4}$$

For fractal domains, the surface integral is regularized by the surface product measure in similarity with the regularization performed along the length as shown in Eq. (1). Thus, after considering the surface projection onto each coordinate plane as illustrated in [14], the surface integral becomes

$$\int_{\partial W_D} f_i n_i \mathrm{d}S_D = \int_{\partial W} f_i c_i^S n_i \mathrm{d}S \tag{5}$$

where $c_i^S = c_V/c_i$. Note that c_i^S is independent of the ith Cartesian coordinate x_i. Applying the conventional Green–Gauss theorem to the right-hand side of Eq. (5), we obtain

$$\int_{\partial W} f_i c_i^S n_i \mathrm{d}S = \int_W (f_i c_i^S)_{,i}\mathrm{d}V = \int_W f_{i,i} c_i^S \mathrm{d}V \tag{6}$$

and using the volume integral regularization as was done in Eq. (2), the volume integral of Eq. (6) is expressed in the fractal configuration as

$$\int_W f_{i,i}c_i^S\mathrm{d}V = \int_{W_D} f_{i,i}\frac{c_i^S}{c_V}\mathrm{d}V_D = \int_{W_D}\frac{1}{c_i}f_{i,i}\mathrm{d}V_D \tag{7}$$