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Nonlinear bi-integrable couplings with Hamiltonian structures

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Abstract

Bi-integrable couplings of soliton equations are presented through introducing non-semisimple matrix Lie algebras on which there exist non-degenerate, symmetric and ad-invariant bilinear forms. The corresponding variational identity yields Hamiltonian structures of the resulting bi-integrable couplings. An application to the AKNS spectral problem gives bi-integrable couplings with Hamiltonian structures for the AKNS equations.

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1. Introduction

Zero curvature equations associated with semisimple Lie algebras generate typical soliton equations such as the Korteweg-de Vries equation, the nonlinear Schrödinger equation and the Kadomtsev–Petviashvili equation [2]. In the case of non-semisimple Lie algebras, zero curvature equations result in integrable couplings of soliton equations [23,24], and the perturbation equations generalizing the symmetry equations are examples of integrable couplings [20,9]. There are very rich mathematical structures behind integrable couplings [20,9,36,31,34] and the study of integrable couplings provide clues towards complete classification of multicomponent integrable equations [20,9,10].

The variational identity on general loop algebras presents Hamiltonian structures for the associated integrable couplings [18,13,16]. Based on special semi-direct sums of Lie algebras, Lax pairs of block form and with several spectral parameters bring diverse interesting integrable couplings with Hamiltonian structures [9,28,29]. A key to construct Hamiltonian structures by the variational identity is the existence of non-degenerate, symmetric and ad-invariant bilinear forms on the underlying Lie algebras.

Let us consider an integrable evolution equation

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots), \tag{1.1}$$

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where u is a column vector of dependent variables. Assume that it has a zero curvature representation [3]:

$$U_t - V_x + [U, V] = 0, \tag{1.2}$$

where the Lax pair [6], U and V, belongs to a matrix loop algebra, let us say, g, i.e.,

$$U = U(u, \lambda),\ V = V(u, \lambda) \in g,\ \lambda \ - \ \text{spectral parameter}. \tag{1.3}$$

An integrable coupling of Eq. (1.1) (see [20,9] for definition):

$$\overline{u}_t = \overline{K}_1(\overline{u}) = \begin{bmatrix} K(u) \\ S(u, v) \end{bmatrix}, \overline{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{1.4}$$

is called nonlinear, if $S(u, v)$ is nonlinear with respect to the sub-vector v of dependent variables [26,17]. An integrable system of the form

$$\overline{u}_t = \overline{K}(\overline{u}) = \begin{bmatrix} K(u) \\ S_1(u, v) \\ S_2(u, v, w) \end{bmatrix}, \overline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{1.5}$$

is called a bi-integrable coupling of Eq. (1.1). Note that in (1.5), S_1 does not depend on w, and the whole system is of triangular form. In this paper, we would like to explore some mathematical structures of Lie algebras and zero curvature equations, to construct bi-integrable couplings and their Hamiltonian structures by using the variational identity associated with the enlarged Lax pairs.

This paper is organized as follows. In Section 2, a kind of matrix Lie algebras will be introduced. Zero curvature equations on the resulting Lie algebras present bi-integrable couplings of soliton equations. In Section 3, an application to the AKNS soliton hierarchy will be made to generate nonlinear bi-integrable couplings and the corresponding variational identity yields Hamiltonian structures for the obtained integrable couplings. An important step in constructing Hamiltonian structures is to find non-degenerate, symmetric and ad-invariant bilinear forms on the underlying Lie algebras. In the last section, a few of concluding remarks will be given, along with discussion on a particular bi-integrable coupling.

2. Matrix Lie algebras and bi-integrable couplings

Let α be an arbitrary fixed constant, which could be zero. To generate bi-integrable couplings, we introduce a kind of block matrices

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & A_2 \\ 0 & 0 & A_1 \end{bmatrix}, \tag{2.1}$$

where $A_i = A_i(\lambda)$, $1 \le i \le 3$, are square matrices of the same order, depending on the free parameter λ. Obviously, we have the matrix commutator relation

$$[M(A_1, A_2, A_3), M(B_1, B_2, B_3)] = M(C_1, C_2, C_3), \tag{2.2}$$

with

$$\begin{cases} C_1 = [A_1, B_1], \\ C_2 = [A_1, B_2] + [A_2, B_1] + \alpha[A_2, B_2], \\ C_3 = [A_1, B_3] + [A_3, B_1] + [A_2, B_2]. \end{cases} \tag{2.3}$$

This closure property implies that all block matrices defined by (2.1) form a matrix Lie algebra. Such matrix Lie algebras create a basis for us to generate nonlinear Hamiltonian bi-integrable couplings. The block A_1 corresponds to the original integrable equation, and the other two blocks A_2 and A_3 are used to generate the supplementary vector fields S_1 and S_2. The commutator $[A_2, B_2]$ yields nonlinear terms in the resulting bi-integrable couplings.