Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 127 (2016) 189–193
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

NSFD scheme for acoustic propagation with the linearized Euler equations

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Received 11 November 2011; received in revised form 27 March 2013; accepted 24 November 2013
Available online 11 August 2014

Abstract

Our goal is to construct a nonstandard finite difference (NSFD) scheme for the linearized Euler partial differential equations (PDE's) modeling acoustic propagation in one space dimension. Unlike other works on this discretization problem, we formulate it in terms of a single, second-order PDE rather than as two separate first-order equations. The important mathematical features of this scheme are discussed.

Keywords: Linearized Euler equations; Acoustic propagation; Nonstandard finite difference schemes

1. Introduction

Our major task is to construct a nonstandard finite difference (NSFD) discretization for the linearized Euler partial differential equations in one space dimension [1–3]. This work extends the earlier calculations of Goodrich [4] who considered various issues related to the propagation of sound over either long times or long distances. These two topics comprise several of the foundational problems in computational aeroacoustics and, to a good approximation, may be modeled by the linearized Euler partial differential equations (PDE's), which correspond to a first-order, linear hyperbolic system [1]. These equations are [4]

$$u_t + Mu_x + p_x = 0, \tag{1.1}$$

$$p_t + Mp_x + u_x = 0, \tag{1.2}$$

where $p(x, t)$ and $u(x, t)$ are, respectively, the pressure and velocity, and the constant, M is the Mach number [2,3].

It should be noted that from a mathematical perspective, M can take any non-negative value. In particular, the work of Goodrich [4] examines, for numerical purposes, a range of M values from zero to two. However, strictly speaking, the linear approximations only hold for M values small compared to one, i.e., $0 \le M \ll 1$. Our work implicitly makes this assumption.

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http://dx.doi.org/10.1016/j.matcom.2013.11.008

If we, respectively, add and subtract Eqs. (1.1) and (1.2), and define

$$w_1 = u + p, \tag{1.3a}$$

$$w_2 = u - p, \tag{1.3b}$$

then $w_1(x, t)$ and $w_2(x, t)$ satisfy the following PDE's

$$[\partial_t + (1 + M)\partial_x]w_1 = 0, \tag{1.4a}$$

$$[\partial_t - (1 - M)\partial_x]w_2 = 0. \tag{1.4b}$$

Also, a direct calculation shows that both $u(x, t)$ and $p(x, t)$ are also solutions to

$$w_{tt} + 2Mw_{tx} - (1 - M^2)w_{xx} = 0, \tag{1.5}$$

a hyperbolic, second-order, linear PDE. To show this, multiply Eqs. (1.4a) and (1.4b), respectively, by $[\partial_t - (1 - M)\,\partial_x]$ and $[\partial_t + (1 + M)\,\partial_x]$, and simplify the resulting expressions to obtain the result of Eq. (1.5). From Eq. (1.4), we have

$$w(x, t) = w_1(x, t) + w_2(x, t), \tag{1.6}$$

where

$$w_1(x, t) = f[x - (1 + M)t], \tag{1.7a}$$

$$w_2(x, t) = g[x + (1 - M)t], \tag{1.7b}$$

and $f(z)$ and $g(z)$ are arbitrary functions having continuous second derivatives. If the initial conditions are taken to be

$$u(x, 0) = u_0(x), \tag{1.8a}$$

$$p(x, 0) = p_0(x), \tag{1.8b}$$

then

$$\begin{aligned} u(x, t) = {} & \left(\frac{1}{2}\right)\{p_0[x - (1 + M)t] - p_0[x + (1 - M)t]\} \\ & + \left(\frac{1}{2}\right)\{u_0[x - (1 + M)t] + u_0[x + (1 - M)t]\}, \end{aligned} \tag{1.9}$$

$$\begin{aligned} p(x, t) = {} & \left(\frac{1}{2}\right)\{p_0[x - (1 + M)t] + p_0[x + (1 - M)t]\} \\ & + \left(\frac{1}{2}\right)\{u_0[x - (1 + M)t] - u_0[x + (1 - M)t]\}. \end{aligned} \tag{1.10}$$

Note that the expressions given in Eqs. (1.9) and (1.10) are the exact solutions to Eqs. (1.1) and (1.2) for the initial conditions listed in Eq. (1.8).

We now demonstrate how a NSFD discretization can be constructed for Eq. (1.5).

2. NSFD scheme for Eq. (1.5)

To proceed, we first use the following discretizations for (x, t, w):

$$\begin{cases} x \to x_m = (\Delta x)m; & m = \text{positive/negative integers;} \\ t \to t_k = (\Delta t)k; & k = 0, 1, 2, \ldots; \\ w(x, t) \to w_m^k. \end{cases} \tag{2.1}$$