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Original Article

Description of kink evolution by means of particular analytical solutions

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Abstract

It is shown, that particular kink-shaped wave solutions to nonlinear nonintegrable equation may be employed to account for important features of the kink evolution observed in numerical solutions and to check the last solutions without computations. Thus, an exact traveling wave solution may predict boundary conditions suitable for the kink realization in numerics. A quasistationary asymptotic solution may detect an evidence of dispersion at the numerical kink shape, while an asymptotic solution obtained by the multiple scale method, describes variations in the kink amplitude, slope and velocity.

Keywords: Kink; Nonintegrable nonlinear equation; Traveling wave solution; Asymptotic solution; Numerical solution

1. Introduction

Most of non-linear partial differential equations (PDE) admit only particular analytical solutions and mainly in the 1D case [1,6]. Only in some cases some $(3+1)$ exact solutions are found [1,7]. The main problem is to account for the whole evolution of the wave in nonintegrable case. However, numerical solutions of nonlinear PDE are very sensitive to the values of the equation coefficients and to the initial conditions, however, this dependence is unlikely to describe numerically. Also some numerical results look rather unusual, and it is difficult to decide whether they are correct or caused by a defect of a scheme. Sometimes an asymptotic solution may explain the behavior of nonlinear waves, e.g., increase or decrease in the amplitude of a solitary wave [1], similarly the conservation laws may be employed for confirmation of the numerical results [2,13]. Approximate solutions may be constructed for some nonintegrable equations to estimate deviations in the exact solutions caused by inelastic (but almost elastic) collision of the solitary waves [4,8]. Some features of a solution are rather close to those of the integrable one, in particular, arising of a nonlinear wave of permanent shape. The last may be described by a single traveling wave solution which is obtained for many nonintegrable equations. Despite the particular nature of these solutions and specific initial conditions required for their existence, they might be used to obtain conditions required for arising of the solitary waves of permanent shape,

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delocalization of the input, solitary wave selection, etc. [9,11,12]. These conditions are obtained from an analysis of the expressions for the solitary wave amplitude, velocity and width, more important is that they are obtained in an explicit analytical form. The conditions allow us to choose suitable values of the equation coefficients in numerics even when the initial conditions differ from those required for the single traveling wave solution.

Most of the papers mentioned above deal with the bell-shaped solitary waves. Now an attention is paid on the kink-shaped waves behavior. The main ideas are illustrated on a particular example. The equation under study arises to account for shear strain waves $v(x,t)$ in a growth plate of a long bone [10],

$$\rho v_{tt} - (\mu(x)v)_{xx} - \gamma(v^3)_{xx} - \eta v_{xxt} = Q v_{xxxx}, \tag{1}$$

The variations in μ are caused by an ossification resulting in variation from the value of a cartilage, μ_c, to that of a bone, μ_b,

$$\mu = \kappa(x)\mu_c + (1-\kappa(x))\mu_b, \tag{2}$$

The parameter κ accounts for biological processes,

$$\kappa = \frac{1}{2}(1+\tanh(mx)), \tag{3}$$

that follows from an analysis performed in Refs. [5,3].

Eq. (1) possesses nonlinear, dispersive and dissipative terms as well as the term responsible for inhomogeneity. It is known, that nonlinear wave of permanent shape exists as result of some balances between these terms. Thus the kink-shaped may be supported by a balance between nonlinearity and dissipation, the case that is realized at $\mu = const$, $Q = 0$ in Eq. (1). Presence of dispersion may still support the kink but more likely gives rise to oscillations at the kink profile at $\mu = const$, $Q \neq 0$. Finally, presence of inhomogeneity, $\mu \neq const$, $Q = 0$, results in deviations in the kink profile, namely, in its amplitude, slope and velocity.

In this paper all these cases will be considered subsequently. It will be demonstrated how and to what extent an particular exact and asymptotic solutions help to predict and explain evolution of a kink in more general numerical solutions, thus, confirming their validity without computations.

2. Use of an exact solution for a design of numerical solution

In the case $\mu = const$, $Q = 0$, Eq. (1) reads

$$\rho v_{tt} - \mu v_{xx} - \gamma(v^3)_{xx} - \eta v_{xxt} = 0. \tag{4}$$

Its exact traveling kink-shaped wave solutions can be systematically obtained by using an idea of transforming nonlinear PDEs into solvable ODEs [6]. A particular solution is obtained by direct integration to the 1st order equation and using a reference book on integrals,

$$v = \frac{C\exp(k\theta)}{\sqrt{1+B\exp(2k\theta)}}, \tag{5}$$

where $\theta = x - Vt$,

$$B = \frac{C^2\gamma}{\rho V^2 - \mu},\quad k = \frac{\mu - \rho V^2}{\eta V}. \tag{6}$$

The bounded wave solution exists for $B > 0$ that happens for $\gamma > 0$, $V < \sqrt{\mu/\rho}$ or for $\gamma < 0$, $V < \sqrt{\mu/\rho}$. Both cases are shown in Fig. 1. The most important result is that the opposite boundary conditions are needed for kinks at different sign of γ, and the kink cannot vanish at $\theta \to -\infty$ for $\gamma > 0$. This analytical prediction allows us to set correct initial condition in the numerical solution of Eq. (4) shown in Fig. 2. One and the same initial condition is chosen in the form of the profile shown in left part of Fig. 1. The times t for the profiles shown in Fig. 2 are 0, 5, 10, and 15 units. The computations for positive γ at the left plot in Fig. 2 demonstrates stable propagation of the kink-shaped wave of permanent shape and velocity according to the exact traveling wave solution. However, the kink does not evolve from the same input at the right plot in Fig. 2 in simulation of Eq. (4) at negative sign of γ.